

On Shannon entropies in μ -deformed Segal-Bargmann analysis

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Abstract

We consider a μ -deformation of the Segal-Bargmann transform, which is a unitary map from a μ -deformed quantum configuration space onto a μ -deformed quantum phase space (the μ -deformed Segal-Bargmann space). Both of these Hilbert spaces have canonical orthonormal bases. We obtain explicit formulas for the Shannon entropy of some of the elements of these bases. We also consider two reverse log-Sobolev inequalities in the μ -deformed Segal-Bargmann space, which have been proved in a previous work, and show that a certain known coefficient in them is the best possible.

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1 Introduction

The Segal-Bargmann space \mathcal{B}^2 is the holomorphic subspace of the Hilbert space $L^2(\mathbb{C}, d\nu_{\text{Gauss}})$, where $d\nu_{\text{Gauss}}$ is a Gaussian measure. Since \mathcal{B}^2 is closed in $L^2(\mathbb{C}, d\nu_{\text{Gauss}})$, the Segal-Bargmann space is itself a Hilbert space. It is common to think of the Segal-Bargmann space as a quantum phase space, similarly as one thinks of the space $L^2(\mathbb{R}, dx)$ as a quantum configuration space. The so called Bargmann transform $\tilde{B} : L^2(\mathbb{R}, dx) \rightarrow \mathcal{B}^2$ is an isomorphism between these two quantum spaces and *Segal-Bargmann analysis* has to do mainly with the study of operators related to \tilde{B} and spaces of holomorphic functions related to \mathcal{B}^2 . (The beginnings of this mathematical theory date back to the works of Segal [Seg1], [Seg2] and Bargmann [Bar]. The physical theory begins with the work of Fock [F].) The quantum configuration space can be replaced by another unitarily equivalent space, namely $L^2(\mathbb{R}, dg)$, called the *ground state representation*, where dg is another Gaussian measure. In this case, the resulting transform B that maps the ground state representation unitarily onto the Segal-Bargmann space is called the *Segal-Bargmann transform*. In both quantum spaces $L^2(\mathbb{R}, dg)$ and \mathcal{B}^2 there are defined unbounded self-adjoint operators Q (position) and P (momentum), which satisfy the relation $[P, Q] = -iI$, called the *canonical commutation relation* (CCR). The CCR implies the *equations of motion* $i[P, H] = Q$ and $i[Q, H] = -P$, where $H = 2^{-1}(Q^2 + P^2)$ is the Hamiltonian of the harmonic oscillator. In 1950, Wigner [Wig] proved that the converse implication is false by exhibiting a family of unbounded operators, labeled by a parameter $\mu > -1/2$, that satisfy the equations of motion but do not satisfy the CCR. Rosenblum and Marron described explicitly (in [Ros1], [Ros2] and [Marr]) a μ -quantum configuration space $L^2(\mathbb{R}, |x|^{2\mu} dx)$, a μ -Segal-Bargmann space \mathcal{B}_μ^2 , and a μ -Bargmann transform \tilde{B}_μ which is a unitary onto transformation mapping the former Hilbert space to the latter Hilbert space. This theory can be understood as a μ -deformation of standard Segal-Bargmann analysis with the property that if one sets $\mu = 0$ the standard theory is recovered (see [Snt3]). So we will refer to $L^2(\mathbb{R}, |x|^{2\mu} dx)$ and \mathcal{B}_μ^2 , as the “ μ -deformed quantum configuration space” and the “ μ -deformed Segal-Bargmann space”, respectively, and to \tilde{B}_μ as the “ μ -deformed Bargmann transform”. It is easy to obtain explicitly also the “ μ -deformed ground state representation” $L^2(\mathbb{R}, dg_\mu)$ and the “ μ -deformed Segal-Bargmann transform” B_μ , which is a unitary map from $L^2(\mathbb{R}, dg_\mu)$ onto \mathcal{B}_μ^2 .

In his paper [Snt1] the second author put emphasis on the Shannon entropy (to be defined in Section 2) as an important quantity in Segal-Bargmann analysis. More precisely, following [Hir] the second author proved a log-Sobolev inequality, where the entropies of a function $f \in L^2(\mathbb{R}, dg)$ and of its Segal-Bargmann transform $Bf \in \mathcal{B}^2$ are involved. Later in [Snt2], the second author

obtained explicit formulas for the entropy of relevant elements of the Hilbert spaces $L^2(\mathbb{R}, dg)$ and \mathcal{B}^2 , namely, elements of the corresponding canonical basis of these spaces. By denoting by ζ_n , $n = 0, 1, \dots$ the functions of the canonical basis $\{\zeta_n\}_{n=0}^\infty$ of the ground state representation $L^2(\mathbb{R}, dg)$, and by ξ_n , $n = 0, 1, \dots$ the functions of the canonical basis $\{\xi_n\}_{n=0}^\infty$ of Segal-Bargmann space \mathcal{B}^2 , the second author proved in [Snt2] that

$$S_{L^2(\mathbb{C}, d\nu_{\text{Gauss}})}(\xi_n) = n \left(-\gamma + 1 + \frac{1}{2} + \dots + \frac{1}{n} \right) - \log n!, \quad (1.1)$$

$$S_{L^2(\mathbb{R}, dg)}(\zeta_1) = 2 - \log 2 - \gamma, \quad (1.2)$$

where $S_{L^2(\mathbb{C}, d\nu_{\text{Gauss}})}(\xi_n)$ is the entropy of $\xi_n \in \mathcal{B}^2$, $n = 0, 1, \dots$, $S_{L^2(\mathbb{R}, dg)}(\zeta_1)$ is the entropy of $\zeta_1 \in L^2(\mathbb{R}, dg)$, and γ is Euler's constant.

In the context of the μ -deformed theory of Segal-Bargmann analysis, similar results to those in [Snt1] have been recently proven, e.g. log-Sobolev and reverse log-Sobolev inequalities. (See [A-S.1], [A-S.2] and [P-S].) What we want to do in this work is to obtain, for the μ -deformed theory, similar results to those in [Snt2]. That is, we want to obtain explicit formulas for the entropies of the μ -deformed elements ζ_n^μ and ξ_n^μ , $n = 0, 1, \dots$ of the corresponding μ -deformed canonical basis $\{\zeta_n^\mu\}_{n=0}^\infty$ and $\{\xi_n^\mu\}_{n=0}^\infty$ of the μ -deformed Hilbert spaces $L^2(\mathbb{R}, dg_\mu)$ and \mathcal{B}_μ^2 , respectively.

We now outline the content of the work. In Section 2 we give the definitions and notation that will be used throughout the work. In this section we also introduce the μ -deformed Hilbert spaces $L^2(\mathbb{R}, dg_\mu)$ and \mathcal{B}_μ^2 , and their canonical bases as well. In Section 3 we give some preliminary results that will help us to analyze some properties of the sequence of entropies of the functions ξ_n^μ , $n = 0, 1, \dots$. These properties are not explicitly given (in the case $\mu = 0$) in [Snt2], but we give them as a proposition at the end of Section 3. In Section 4 we obtain explicit formulas for the entropies of the elements $\xi_n^\mu \in \mathcal{B}_\mu^2$, $n = 1, 2, \dots$, and we study some properties of the corresponding sequence of entropies. The results in this section generalize the formula (1.1) of [Snt2], as well as the proposition at the end of Section 3 mentioned above. In Section 5 we consider the μ -deformed ground state representation and we obtain explicit formulas for the monomials $t^n \in L^2(\mathbb{R}, dg_\mu)$, $n = 0, 1, \dots$. Unfortunately the technique we use here to obtain these formulas (and those of Section 4) does not work to obtain the entropies of the elements ζ_n^μ of the canonical basis of $L^2(\mathbb{R}, dg_\mu)$, for $n \geq 2$. It turns out that our method for calculating the entropy of a function f works only in the case of f being a monomial, and the elements ζ_n^μ are monomials only for $n = 0, 1$. The formula we obtain for the entropy of ζ_1^μ generalizes the formula (1.2) of [Snt2].

Also, by means of a concrete example, in Section 5 we show that the μ -deformed Segal-Bargmann transform B_μ does not preserve entropy. In Section 6 we consider two reverse log-Sobolev inequalities proved in [A-S.2], in which the condition $c > 1$ of a certain parameter c appears as a sufficient condition. In this section we show that this condition is also necessary, or in other words, that the condition $c > 1$ is the best possible. Finally, in Section 7 we make some

comments about what we left unfinished in this paper and what is possible to do beyond the results presented here.

2 Definitions and notation

In this section we give the definitions and the notation that we will use throughout the work. First, we take $\mu > -\frac{1}{2}$ to be a fixed parameter (unless otherwise stated). The (Coxeter) group \mathbb{Z}_2 is the multiplicative group $\{-1, 1\}$, and \log is the natural logarithm (base e). We use the convention $0 \log 0 = 0$ (which makes the function $\phi : [0, \infty) \rightarrow \mathbb{R}$, $\phi(x) = x \log x$ continuous). We also use the convention that C denotes a constant (a quantity that does not depend on the variables of interest in the context), which may change its value every time it appears. We denote by $\mathcal{H}(\mathbb{C})$ the space of holomorphic functions $f : \mathbb{C} \rightarrow \mathbb{C}$ with the topology of uniform convergence on compact sets.

We begin by defining the μ -deformations of the factorial function and of the exponential function. Let \mathbb{N} denote the set of positive integers.

Definition 2.1 *The μ -deformed factorial function $\gamma_\mu : \mathbb{N} \cup \{0\} \rightarrow \mathbb{R}$ is defined by $\gamma_\mu(0) = 1$ and*

$$\gamma_\mu(n) := (n + 2\mu\theta(n)) \gamma_\mu(n-1),$$

where $n \in \mathbb{N}$ and $\theta : \mathbb{N} \rightarrow \{0, 1\}$ is the characteristic function of the odd positive integers. The μ -deformed exponential function $\mathbf{e}_\mu : \mathbb{C} \rightarrow \mathbb{C}$, is defined by the power series

$$\mathbf{e}_\mu(z) := \sum_{n=0}^{\infty} \frac{z^n}{\gamma_\mu(n)}.$$

We note that $\gamma_0(n) = n!$ (the usual factorial function) and so $\mathbf{e}_0(z) = \exp(z)$ (the usual complex exponential function). It is clear that the power series in the definition of $\mathbf{e}_\mu(z)$ is absolutely convergent for all $z \in \mathbb{C}$. So the μ -deformed exponential \mathbf{e}_μ is an entire function.

We will use the following explicit formulas for $\gamma_\mu(2n)$ and $\gamma_\mu(2n+1)$, $n = 0, 1, 2, \dots$ (see [Ros1], p. 371):

$$\begin{aligned} \gamma_\mu(2n) &= \frac{2^{2n} \Gamma(n+1) \Gamma(\mu + n + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \\ &= \frac{(2n)! \Gamma(\frac{1}{2}) \Gamma(\mu + n + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2}) \Gamma(n + \frac{1}{2})}, \end{aligned} \tag{2.1}$$

$$\begin{aligned}\gamma_\mu(2n+1) &= \frac{2^{2n+1}\Gamma(n+1)\Gamma(\mu+n+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})} \\ &= \frac{(2n+1)!\Gamma(\frac{1}{2})\Gamma(\mu+n+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})\Gamma(n+\frac{3}{2})}.\end{aligned}\tag{2.2}$$

The following definition (from [Ros1]) gives us a μ -deformation of the classical Hermite polynomials.

Definition 2.2 For $n = 0, 1, \dots$ we define the n -th μ -deformed Hermite polynomial $H_n^\mu(t)$ by the generating function

$$\exp(-z^2) \mathbf{e}_\mu(2tz) = \sum_{n=0}^{\infty} H_n^\mu(t) \frac{z^n}{n!}.$$

It is easy to check that $H_n^\mu(t)$ is in fact a polynomial of degree n in the real variable t . For example, we have that $H_0^\mu(t) = 1$, $H_1^\mu(t) = \frac{2}{1+2\mu}t$, $H_2^\mu(t) = \frac{4}{1+2\mu}t^2 - 2$, and so on.

The normalized μ -deformed Hermite polynomials $\zeta_n^\mu(t)$, $n = 0, 1, \dots$ defined by

$$\zeta_n^\mu(t) := 2^{-\frac{n}{2}} (n!)^{-1} (\gamma_\mu(n))^{\frac{1}{2}} H_n^\mu(t),\tag{2.3}$$

form an orthonormal basis of the μ -deformed ground state representation $L^2(\mathbb{R}, dg_\mu)$, where dg_μ is the μ -deformed Gaussian measure defined by

$$dg_\mu(t) := \left(\Gamma\left(\mu + \frac{1}{2}\right) \right)^{-1} \exp(-t^2) |t|^{2\mu} dt.\tag{2.4}$$

The basis $\{\zeta_n^\mu\}_{n=0}^\infty$ is called *the canonical basis of $L^2(\mathbb{R}, dg_\mu)$* . (See [Ros1] and [P-S].)

The case $\mu = 0$ recovers the well known fact that for $n = 0, 1, \dots$, the normalized polynomials $\zeta_n(t) = 2^{-\frac{n}{2}} (n!)^{-\frac{1}{2}} H_n(t)$, where $H_n(t)$ denotes the n -th Hermite polynomial, form the canonical orthonormal basis of the ground state representation $L^2(\mathbb{R}, dg)$, where dg is the Gaussian probability measure $dg(t) = \pi^{-\frac{1}{2}} \exp(-t^2) dt$. (See [Hall].)

Definition 2.3 We define the measure $d\nu_\mu$ on the space $\mathbb{C} \times \mathbb{Z}_2$ by

$$d\nu_\mu(z, 1) := \frac{2^{\frac{1}{2}-\mu}}{\pi\Gamma(\mu+\frac{1}{2})} K_{\mu-\frac{1}{2}}(|z|^2) |z|^{2\mu+1} dx dy,\tag{2.5}$$

$$d\nu_\mu(z, -1) := \frac{2^{\frac{1}{2}-\mu}}{\pi\Gamma(\mu+\frac{1}{2})} K_{\mu+\frac{1}{2}}(|z|^2) |z|^{2\mu+1} dx dy,\tag{2.6}$$

where Γ is the Euler gamma function, K_α is the Macdonald function of order α (both defined in [Leb]), and $dxdy$ is Lebesgue measure on \mathbb{C} .

By using that $\mathbb{C} \cong \mathbb{C} \times \{1\} \cong \mathbb{C} \times \{-1\}$, we will identify the restrictions (2.5) and (2.6) as measures on \mathbb{C} .

The Macdonald function K_α is the modified Bessel function of the third kind (with purely imaginary argument, as described in [Wat], p. 78), which is known to be a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$ and is entire with respect to the parameter α . Nevertheless, our interest will be only in the values and behavior of this function for $x \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$. For $z \in \mathbb{C}$, $|\arg z| < \pi$ and $\alpha \notin \mathbb{Z}$, the Macdonald function can be defined as

$$K_\alpha(z) = \frac{\pi}{2} \frac{I_{-\alpha}(z) - I_\alpha(z)}{\sin(\alpha\pi)}$$

(see [Leb], p. 108), where $I_\alpha(z)$ is the modified Bessel function of the first kind. For $\alpha \in \mathbb{Z}$, we define $K_\alpha(z) = \lim_{\beta \rightarrow \alpha} K_\beta(z)$. This expression shows that $K_\alpha(z)$ is an even function of the parameter α . In particular, since $I_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sinh z$ and $I_{-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cosh z$ (see [Leb], p. 112), we have that

$$K_{\pm\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \exp(-z),$$

which shows that for $\mu = 0$ the measures defined on \mathbb{C} by (2.5) and (2.6) are the same Gaussian measure:

$$d\nu_0(z, 1) = d\nu_0(z, -1) = \pi^{-1} \exp(-|z|^2) dxdy,$$

which is the Gaussian measure $d\nu_{\text{Gauss}}$ of the Segal-Bargmann space $\mathcal{B}^2 = \mathcal{H}(\mathbb{C}) \cap L^2(\mathbb{C}, d\nu_{\text{Gauss}})$.

By using the formula

$$\int_0^\infty K_\alpha(s) s^{\beta-1} ds = 2^{\beta-2} \Gamma\left(\frac{\beta-\alpha}{2}\right) \Gamma\left(\frac{\beta+\alpha}{2}\right), \quad (2.7)$$

which holds if $\text{Re } \beta > |\text{Re } \alpha|$ (see [Wat], p. 388), we can see that (2.5) and (2.6) are finite measures on \mathbb{C} , and moreover that the former is a probability measure. (See [P-S].)

The integral representation

$$K_\alpha(z) = \int_0^\infty \exp(-z \cosh u) \cosh(\alpha u) du \quad \text{Re } z > 0 \quad (2.8)$$

(see [Leb], p. 119) gives us at once two important properties of the Macdonald function. The first is that $K_\alpha(x) > 0$ for all $x \in \mathbb{R}^+$, and the second is that K_α is a monotone decreasing function for $x \in \mathbb{R}^+$.

We will work with the Hilbert space $L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_\mu)$. The norm of a vector $f \in L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_\mu)$ will be denoted by $\|f\|_{L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_\mu)}$. Let us consider the space

$$\mathfrak{H}_{2,\mu} = \left\{ f : \mathbb{C} \rightarrow \mathbb{C} \mid f_e \in L^2(\mathbb{C}, d\nu_\mu|_{\mathbb{C} \times \{1\}}) \text{ and } f_o \in L^2(\mathbb{C}, d\nu_\mu|_{\mathbb{C} \times \{-1\}}) \right\},$$

where $f = f_e + f_o$ is the decomposition of f into its even and odd parts. Observe that when $\mu = 0$ we have $\mathfrak{H}_{2,0} = L^2(\mathbb{C}, d\nu_{\text{Gauss}})$.

For $f \in \mathfrak{H}_{2,\mu}$ we define

$$\|f\|_{\mathfrak{H}_{2,\mu}}^2 := \|f_e\|_{L^2(\mathbb{C}, d\nu_\mu|_{\mathbb{C} \times \{1\}})}^2 + \|f_o\|_{L^2(\mathbb{C}, d\nu_\mu|_{\mathbb{C} \times \{-1\}})}^2.$$

The linear map $\Phi : \mathfrak{H}_{2,\mu} \rightarrow L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_\mu)$ defined as $(\Phi f)(z, 1) = f_e(z)$ and $(\Phi f)(z, -1) = f_o(z)$ is injective and has the property that

$$\|f\|_{\mathfrak{H}_{2,\mu}} = \|\Phi f\|_{L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_\mu)} \quad (2.9)$$

for all $f \in \mathfrak{H}_{2,\mu}$. Therefore $\|\cdot\|_{\mathfrak{H}_{2,\mu}}$ is a norm on $\mathfrak{H}_{2,\mu}$. It is not hard to show that the range of Φ is a closed subspace of $L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_\mu)$. Therefore $\mathfrak{H}_{2,\mu}$ is a Hilbert space, since we have identified it with a closed subspace of the Hilbert space $L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_\mu)$. For a function $f \in \mathfrak{H}_{2,\mu}$ we will sometimes write its norm as $\|f\|_{L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_\mu)}$, meaning that we are using (2.9) and identifying f with Φf .

We will use the notations $d\nu_{e,\mu}$ and $d\nu_{o,\mu}$ for the restrictions $d\nu_\mu|_{\mathbb{C} \times \{1\}}$ and $d\nu_\mu|_{\mathbb{C} \times \{-1\}}$, respectively. So for $f \in \mathfrak{H}_{2,\mu}$ we have

$$\begin{aligned} \|f\|_{\mathfrak{H}_{2,\mu}}^2 &= \|f_e\|_{L^2(\mathbb{C}, d\nu_{e,\mu})}^2 + \|f_o\|_{L^2(\mathbb{C}, d\nu_{o,\mu})}^2 \\ &= \|f_e\|_{\mathfrak{H}_{2,\mu}}^2 + \|f_o\|_{\mathfrak{H}_{2,\mu}}^2. \end{aligned}$$

Definition 2.4 *The μ -deformed Segal-Bargmann space, denoted by \mathcal{B}_μ^2 , is defined as*

$$\mathcal{B}_\mu^2 := \mathcal{H}(\mathbb{C}) \cap \mathfrak{H}_{2,\mu}. \quad (2.10)$$

That is, \mathcal{B}_μ^2 is the holomorphic subspace of $\mathfrak{H}_{2,\mu}$. It turns out that \mathcal{B}_μ^2 is closed in $\mathfrak{H}_{2,\mu}$, and then it is also closed in $L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_\mu)$, so \mathcal{B}_μ^2 is itself a Hilbert space. (The proof of this fact does not depend on μ ; see Theorem 2.2 in [Hall] for the case $\mu = 0$.) Observe that when $\mu = 0$ we have $\mathcal{B}_0^2 = \mathcal{H}(\mathbb{C}) \cap \mathfrak{H}_{2,0} = \mathcal{H}(\mathbb{C}) \cap L^2(\mathbb{C}, d\nu_{\text{Gauss}}) = \mathcal{B}^2$.

If we decompose the space $\mathcal{H}(\mathbb{C})$ of holomorphic functions $f : \mathbb{C} \rightarrow \mathbb{C}$ as $\mathcal{H}(\mathbb{C}) = \mathcal{H}_e(\mathbb{C}) \oplus \mathcal{H}_o(\mathbb{C})$, where

$$\begin{aligned} \mathcal{H}_e(\mathbb{C}) &:= \{f \in \mathcal{H}(\mathbb{C}) : f = f_e\} \\ \text{and } \mathcal{H}_o(\mathbb{C}) &:= \{f \in \mathcal{H}(\mathbb{C}) : f = f_o\} \end{aligned}$$

are the subspaces of the even and odd functions of $\mathcal{H}(\mathbb{C})$, respectively, then by writing $\mathcal{H}(\mathbb{C}) \ni f = f_e + f_o$, the space \mathcal{B}_μ^2 is just the space of holomorphic functions $f : \mathbb{C} \rightarrow \mathbb{C}$ such that the even part f_e (the odd part f_o) of f is square integrable with respect to the measure $d\nu_{e,\mu}$ (with respect to the measure $d\nu_{o,\mu}$, respectively). That is,

$$\mathcal{B}_\mu^2 = \{f \in \mathcal{H}(\mathbb{C}) : f_e \in L^2(\mathbb{C}, d\nu_{e,\mu}) \text{ and } f_o \in L^2(\mathbb{C}, d\nu_{o,\mu})\}.$$

Yet another way to think of \mathcal{B}_μ^2 is as

$$\mathcal{B}_\mu^2 = \mathcal{B}_{e,\mu}^2 \oplus \mathcal{B}_{o,\mu}^2, \quad (2.11)$$

where

$$\begin{aligned} \mathcal{B}_{e,\mu}^2 &= \mathcal{H}_e(\mathbb{C}) \cap \mathfrak{H}_{2,\mu} \\ \text{and } \mathcal{B}_{o,\mu}^2 &= \mathcal{H}_o(\mathbb{C}) \cap \mathfrak{H}_{2,\mu} \end{aligned}$$

are the even and odd subspaces of \mathcal{B}_μ^2 .

Observe that the inner product of the Hilbert space \mathcal{B}_μ^2 (from which the norm on \mathcal{B}_μ^2 defined above comes) is

$$\langle f, g \rangle_{\mathcal{B}_\mu^2} = \langle f_e, g_e \rangle_{L^2(\mathbb{C}, d\nu_{e,\mu})} + \langle f_o, g_o \rangle_{L^2(\mathbb{C}, d\nu_{o,\mu})}. \quad (2.12)$$

We then have that $\mathcal{B}_{e,\mu}^2$ and $\mathcal{B}_{o,\mu}^2$ are orthogonal subspaces of \mathcal{B}_μ^2 , and that (2.11) holds as Hilbert spaces.

The monomials $\xi_n^\mu(z)$, $n = 0, 1, \dots$ defined for $z \in \mathbb{C}$ by

$$\xi_n^\mu(z) := (\gamma_\mu(n))^{-\frac{1}{2}} z^n, \quad (2.13)$$

form an orthonormal basis of the μ -deformed Segal-Bargmann space \mathcal{B}_μ^2 . The basis $\{\xi_n^\mu\}_{n=0}^\infty$ is called the *canonical basis of \mathcal{B}_μ^2* . When $\mu = 0$ we obtain the monomials $\xi_n(z) = (n!)^{-\frac{1}{2}} z^n$, $n = 0, 1, \dots$ which are known to form the canonical basis of the Segal-Bargmann space \mathcal{B}^2 . (See [Hall].)

The μ -deformed Segal-Bargmann transform $B_\mu : L^2(\mathbb{R}, dg_\mu) \rightarrow \mathcal{B}_\mu^2$ can be defined as $B_\mu(\zeta_n^\mu) = \xi_n^\mu$, $n = 0, 1, \dots$. It is clear that B_μ so defined is a unitary map. An explicit formula for B_μ is

$$(B_\mu f)(z) = \exp\left(-\frac{z^2}{2}\right) \int_{\mathbb{R}} \mathbf{e}_\mu\left(2^{\frac{1}{2}}tz\right) f(t) dg_\mu(t). \quad (2.14)$$

(See [P-S].) When $\mu = 0$ this formula becomes

$$(B_0 f)(z) = \int_{\mathbb{R}} \exp\left(-\frac{z^2}{2} + 2^{\frac{1}{2}}tz\right) f(t) dg(t),$$

which is the undeformed Segal-Bargmann transform studied, for example, in [Hall], where it is shown that it is a unitary map from the quantum configuration space $L^2(\mathbb{R}, dg)$ onto the quantum phase space \mathcal{B}^2 .

Definition 2.5 Let $(\Omega, d\nu)$ be a finite measure space, that is, $0 < \nu(\Omega) < \infty$. For $f \in L^2(\Omega, d\nu)$, the Shannon entropy $S_{L^2(\Omega, d\nu)}(f)$ is defined by

$$S_{L^2(\Omega, d\nu)}(f) := \int_{\Omega} |f(\omega)|^2 \log |f(\omega)|^2 d\nu(\omega) - \|f\|_{L^2(\Omega, d\nu)}^2 \log \|f\|_{L^2(\Omega, d\nu)}^2. \quad (2.15)$$

This definition was introduced by Shannon [Sha] in his Theory of Communication. Note that, since $(\Omega, d\nu)$ is a finite measure space, the entropy $S_{L^2(\Omega, d\nu)}(f)$ makes sense for all $f \in L^2(\Omega, d\nu)$. Moreover, by considering the convex function $\phi : [0, \infty) \rightarrow \mathbb{R}$, $\phi(x) = x \log x$, and the probability measure space $(\Omega, d\nu')$, where $d\nu' = W^{-1}d\nu$, $W = \nu(\Omega)$, we have by Jensen's inequality (see [L-L], p. 38) that

$$\left(\int_{\Omega} |f(\omega)|^2 d\nu(\omega) \right) \log \left(\frac{1}{W} \int_{\Omega} |f(\omega)|^2 d\nu(\omega) \right) \leq \int_{\Omega} |f(\omega)|^2 \log |f(\omega)|^2 d\nu(\omega)$$

or

$$(-\log W) \|f\|_{L^2(\Omega, d\nu)}^2 \leq S_{L^2(\Omega, d\nu)}(f),$$

which shows that $S_{L^2(\Omega, d\nu)}(f) \neq -\infty$, though $S_{L^2(\Omega, d\nu)}(f) = +\infty$ can happen. Also observe that $S_{L^2(\Omega, d\nu)}(f) \geq 0$, though $S_{L^2(\Omega, d\nu)}(f)$ can be negative. Finally, note that $S_{L^2(\Omega, d\nu)}(f)$ is homogeneous of degree 2.

Observe that for $f \in \mathcal{B}_{\mu}^2$, $f \neq 0$, the entropy $S_{L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_{\mu})}(f)$ is *not* in general equal to $S_{L^2(\mathbb{C}, d\nu_{e, \mu})}(f_e) + S_{L^2(\mathbb{C}, d\nu_{o, \mu})}(f_o)$. What we really have is

$$\begin{aligned} S_{L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_{\mu})}(f) &= S_{L^2(\mathbb{C}, d\nu_{e, \mu})}(f_e) + S_{L^2(\mathbb{C}, d\nu_{o, \mu})}(f_o) \\ &\quad + \|f_e\|_{L^2(\mathbb{C}, d\nu_{e, \mu})}^2 \log \frac{\|f_e\|_{L^2(\mathbb{C}, d\nu_{e, \mu})}^2}{\|f\|_{L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_{\mu})}^2} \\ &\quad + \|f_o\|_{L^2(\mathbb{C}, d\nu_{o, \mu})}^2 \log \frac{\|f_o\|_{L^2(\mathbb{C}, d\nu_{o, \mu})}^2}{\|f\|_{L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_{\mu})}^2}. \end{aligned} \quad (2.16)$$

Nevertheless, observe that if f is an even (odd) function, its entropy is given by $S_{L^2(\mathbb{C}, d\nu_{e, \mu})}(f)$ ($S_{L^2(\mathbb{C}, d\nu_{o, \mu})}(f)$, respectively). Then, for the functions ξ_n^{μ} of the canonical basis of \mathcal{B}_{μ}^2 we have $S_n^{\mu} = S_{L^2(\mathbb{C}, d\nu_{e, \mu})}(\xi_n^{\mu})$ if n is even, and $S_n^{\mu} = S_{L^2(\mathbb{C}, d\nu_{o, \mu})}(\xi_n^{\mu})$ if n is odd, where $S_n^{\mu} := S_{L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_{\mu})}(\xi_n^{\mu})$, $n = 0, 1, 2, \dots$

3 Preliminary results

In the calculations we will do in the Sections 4 and 5, the derivative of the gamma function will arise naturally. Recall that the *logarithmic derivative* of $z \mapsto \Gamma(z)$, also called the *digamma function* and denoted by $\psi(z)$, is defined by

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}$$

for all $z \neq 0, -1, -2, \dots$ (See [Leb], p. 5.) We will be interested only in the values and behavior of $\psi(x)$ with $x \in \mathbb{R}^+$.

From the basic property of the gamma function $\Gamma(x+1) = x\Gamma(x)$ one obtains the formula

$$\psi(x+1) = \frac{1}{x} + \psi(x),$$

from which one gets by induction that

$$\psi(x+n) = \sum_{k=0}^{n-1} \frac{1}{x+k} + \psi(x)$$

for $n \in \mathbb{N}$. Using the identities $\psi(1) = -\gamma$ and $\psi(\frac{1}{2}) = -\gamma - 2\log 2$ (see [Leb], p. 6), the previous formula implies (by taking $x = 1$ and $x = \frac{1}{2}$) that

$$\psi(n+1) = -\gamma + \sum_{k=1}^n \frac{1}{k}$$

and

$$\psi\left(n + \frac{1}{2}\right) = -\gamma - 2\log 2 + 2 \sum_{k=1}^n \frac{1}{2k-1},$$

When necessary we will use these formulas without further comment.

In this section we will state and prove two lemmas that we will be using in Sections 4 and 5.

Lemma 3.1 (a) *The inequality $0 < \psi(x+m) - \log x < (2m-1)(2x)^{-1}$ holds for all $x \in \mathbb{R}^+$ and $m \in \mathbb{N}$. In particular, we have that for any $m \in \mathbb{N}$*

$$\lim_{x \rightarrow +\infty} (\psi(x+m) - \log x) = 0.$$

(b) *For $y > 0$ fixed we have that*

$$\lim_{x \rightarrow +\infty} (\psi(x+y) - \log x) = 0.$$

(c) *The inequality $-x^{-1} < \psi(x) - \log x < -(2x)^{-1}$ holds for all $x \in \mathbb{R}^+$. In particular, we have that*

$$\lim_{x \rightarrow +\infty} (\psi(x) - \log x) = 0.$$

Proof: From the integral representation of $\psi(z)$,

$$\psi(z) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-tz}}{1-e^{-t}} \right) dt,$$

and the integral representation of $\log(z)$,

$$\log(z) = \int_0^\infty \frac{e^{-t} - e^{-tz}}{t} dt,$$

both valid for $\operatorname{Re} z > 0$ (see [Leb], pp. 6,7), one obtains for all $x > 0$ and $m > 0$ that

$$\psi(x+m) - \log x = \int_0^\infty \left(\frac{1}{t} - \frac{e^{-tm}}{1-e^{-t}} \right) e^{-tx} dt. \quad (3.1)$$

For $m \in \mathbb{N}$, let us consider the function $h_m : \mathbb{R} \rightarrow \mathbb{R}$,

$$h_m(t) = \frac{1}{t} - \frac{e^{-tm}}{1-e^{-t}},$$

where we define $h_m(0) = \lim_{t \rightarrow 0} h_m(t) = \frac{2m-1}{2} > 0$. So h_m is continuous. For all $t > 0$ we will prove by induction that $0 < h_m(t) < \frac{2m-1}{2}$ holds for all $m \in \mathbb{N}$. Observe that $e^t > 1+t$ for $t > 0$ implies $h_1(t) > 0$ for $t > 0$. Also observe that $\beta(t) = \tanh \frac{t}{2} - \frac{t}{2}$ is a decreasing function in \mathbb{R}^+ , so that $\tanh \frac{t}{2} < \frac{t}{2}$ for $t > 0$, which implies that $h_1(t) < \frac{1}{2}$ for $t > 0$. This proves the inequality $0 < h_m(t) < \frac{2m-1}{2}$ for $m = 1$. Suppose now that the inequality holds for a given $m \in \mathbb{N}$. The hypothesis $h_m(t) > 0$ gives us

$$h_{m+1}(t) = \frac{1}{t} - \frac{e^{-tm}}{1-e^{-t}} e^{-t} = \left(\frac{1}{t} - \frac{e^{-tm}}{1-e^{-t}} \right) e^{-t} + \frac{1-e^{-t}}{t} > 0$$

for $t > 0$. Also, the case $m = 1$ gives us that $\frac{1}{t} < \frac{1}{2} + \frac{e^{-t}}{1-e^{-t}}$, which together with the hypothesis $h_m(t) < \frac{2m-1}{2}$ gives us (for $t > 0$) that

$$\begin{aligned} h_{m+1}(t) &= \left(\frac{1}{t} - \frac{e^{-tm}}{1-e^{-t}} \right) e^{-t} + \frac{1-e^{-t}}{t} \\ &< \frac{2m-1}{2} e^{-t} + (1-e^{-t}) \left(\frac{1}{2} + \frac{e^{-t}}{1-e^{-t}} \right) \\ &= \frac{2m-1}{2} e^{-t} + \frac{1+e^{-t}}{2} \\ &= me^{-t} + \frac{1}{2} \\ &< \frac{2m+1}{2}, \end{aligned}$$

as wanted. Then (3.1) and the inequality $0 < h_m(t) < \frac{2m-1}{2}$ we just proved above gives us that

$$0 < \psi(x+m) - \log x < \frac{2m-1}{2} \int_0^\infty e^{-tx} dt = (2m-1)(2x)^{-1},$$

which proves (a).

For $x \in \mathbb{R}^+$ we have that

$$\psi(x) - \log(x) = \psi(x+1) - \log(x) - x^{-1}.$$

So, by using (a) with $m = 1$ we have that

$$-x^{-1} < \psi(x) - \log(x) < (2x)^{-1} - x^{-1} = -(2x)^{-1},$$

which proves (c).

Now we prove (b). (We need to prove the result for $y \notin \mathbb{N}$.) Observe that it is sufficient to demonstrate the result for $y \in (0, 1)$, since given that for any fixed non-integer $Y > 0$ we can write $Y = \lfloor Y \rfloor + y$, where $\lfloor Y \rfloor$ is the floor function of Y and $y \in (0, 1)$. Then, by defining $X := x + \lfloor Y \rfloor$ we have that

$$\begin{aligned} \lim_{x \rightarrow +\infty} (\psi(x + Y) - \log x) &= \lim_{X \rightarrow +\infty} (\psi(X + y) - \log(X - \lfloor Y \rfloor)) \\ &= \lim_{X \rightarrow +\infty} \left(\psi(X + y) - \log X - \log \frac{X - \lfloor Y \rfloor}{X} \right) \\ &= \lim_{X \rightarrow +\infty} (\psi(X + y) - \log X) \\ &= 0. \end{aligned}$$

We consider the continuous function $h_y : \mathbb{R} \rightarrow \mathbb{R}$, $h_y(t) = \frac{1}{t} - \frac{e^{-ty}}{1-e^{-t}}$, where $h_y(0) = \lim_{t \rightarrow 0} h_y(t) = \frac{2y-1}{2}$, and $0 < y < 1$ is fixed. According to (3.1), with $m = y \in (0, 1)$, it is sufficient to prove that h_y is bounded in $[0, \infty)$, since if $|h_y(t)| \leq C$ for all $t \geq 0$, then

$$|\psi(x + y) - \log x| = \left| \int_0^\infty h_y(t) e^{-tx} dt \right| \leq C \int_0^\infty e^{-tx} dt = \frac{C}{x},$$

and thus $\psi(x + y) - \log x \rightarrow 0$ as $x \rightarrow +\infty$. But observe that $\lim_{t \rightarrow +\infty} h_y(t) = 0$ and that h_y is continuous, which shows that h_y is bounded on $[0, \infty)$.

Q.E.D.

Lemma 3.2 *Let $\mu > -\frac{1}{2}$ be fixed. Then*

$$\lim_{n \rightarrow \infty} \frac{(\gamma_\mu(n))^{\frac{1}{n}}}{n} = e^{-1}.$$

(Note that this limit does not depend on μ .)

Proof: It is sufficient to prove that

$$\lim_{n \rightarrow \infty} \frac{(\gamma_\mu(2n))^{\frac{1}{2n}}}{2n} = \lim_{n \rightarrow \infty} \frac{(\gamma_\mu(2n+1))^{\frac{1}{2n+1}}}{2n+1} = e^{-1}.$$

Let us consider the even case. We can write by using formula (2.1) that

$$\frac{(\gamma_\mu(2n))^{\frac{1}{2n}}}{2n} = \frac{((2n)!)^{\frac{1}{2n}}}{2n} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \right)^{\frac{1}{2n}} \left(\frac{\Gamma(\mu + n + \frac{1}{2})}{\Gamma(n + \frac{1}{2})} \right)^{\frac{1}{2n}}.$$

We have that $\lim_{n \rightarrow \infty} \frac{((2n)!)^{\frac{1}{2n}}}{2n} = e^{-1}$ and $\lim_{n \rightarrow \infty} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \right)^{\frac{1}{2n}} = 1$. So it remains to prove that the limit of the third factor in the left hand side is 1. By

using Stirling's formula we have that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \left(\frac{\Gamma(\mu + n + \frac{1}{2})}{\Gamma(n + \frac{1}{2})} \right)^{\frac{1}{2n}} &= \lim_{n \rightarrow \infty} \left(\frac{\sqrt{2\pi} (\mu + n + \frac{1}{2})^{\mu+n} e^{-(\mu+n+\frac{1}{2})}}{\sqrt{2\pi} (n + \frac{1}{2})^n e^{-(n+\frac{1}{2})}} \right)^{\frac{1}{2n}} \\
&= \lim_{n \rightarrow \infty} \left(\left(\mu + n + \frac{1}{2} \right)^{\frac{\mu}{2n}} e^{-\frac{\mu}{2n}} \left(\frac{\mu + n + \frac{1}{2}}{n + \frac{1}{2}} \right)^{\frac{1}{2}} \right) \\
&= 1.
\end{aligned}$$

For the odd case, by using (2.2) we have that

$$\frac{(\gamma_\mu(2n+1))^{\frac{1}{2n+1}}}{2n+1} = \frac{((2n+1)!)^{\frac{1}{2n+1}}}{2n+1} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \right)^{\frac{1}{2n+1}} \left(\frac{\Gamma(\mu + n + \frac{3}{2})}{\Gamma(n + \frac{3}{2})} \right)^{\frac{1}{2n+1}}.$$

We have that $\lim_{n \rightarrow \infty} \frac{((2n+1)!)^{\frac{1}{2n+1}}}{2n+1} = e^{-1}$ and $\lim_{n \rightarrow \infty} \left(\frac{\Gamma(\frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \right)^{\frac{1}{2n+1}} = 1$.

So the proof ends by showing that the limit of the third factor in the left hand side is 1. By using Stirling's formula we have that

$$\begin{aligned}
&\lim_{n \rightarrow \infty} \left(\frac{\Gamma(\mu + n + \frac{3}{2})}{\Gamma(n + \frac{3}{2})} \right)^{\frac{1}{2n+1}} \\
&= \lim_{n \rightarrow \infty} \left(\frac{\sqrt{2\pi} (\mu + n + \frac{3}{2})^{\mu+n+1} e^{-(\mu+n+\frac{3}{2})}}{\sqrt{2\pi} (n + \frac{3}{2})^{n+1} e^{-(n+\frac{3}{2})}} \right)^{\frac{1}{2n+1}} \\
&= \lim_{n \rightarrow \infty} \left(\left(\mu + n + \frac{3}{2} \right)^{\frac{\mu}{2n+1}} e^{-\frac{\mu}{2n+1}} \left(\frac{\mu + n + \frac{3}{2}}{n + \frac{3}{2}} \right)^{\frac{n+1}{2n+1}} \right) \\
&= 1.
\end{aligned}$$

Q.E.D.

Observe that formula (1.1), which gives us the entropy of the elements of the canonical basis $\{\xi_n\}$ of \mathcal{B}^2 , can be written as

$$S_{L^2(\mathbb{C}, d\nu_{\text{Gauss}})}(\xi_n) = n\psi(n+1) - \log n!. \quad (3.2)$$

In the case $n = 0$ we have $\xi_0 = 1$ and then from (2.15) we have that $S_{L^2(\mathbb{C}, d\nu_{\text{Gauss}})}(1) = 0$. (Note that this case is also included in (3.2).)

We can use Lemmas 3.1 and 3.2 to prove some properties of the sequence of

entropies $\{S_n\}_{n=0}^\infty$, where $S_n := S_{L^2(\mathbb{C}, d\nu_{\text{Gauss}})}(\xi_n)$. First, we note that

$$\begin{aligned} S_{n+1} &= (n+1) \psi(n+2) - \log(n+1)! \\ &= (n+1) \left(\frac{1}{n+1} + \psi(n+1) \right) - \log n! - \log(n+1) \\ &= S_n + 1 + \psi(n+1) - \log(n+1) \\ &> S_n + 1 - \frac{1}{n+1}, \end{aligned}$$

where we used Lemma 3.1 (c). Thus, for $n = 0$ we have that $S_1 > 0$, and for $n \in \mathbb{N}$ we have $S_{n+1} > S_n$. That is, the sequence $\{S_n\}_{n=0}^\infty$ is increasing. Moreover, $\{S_n\}_{n=0}^\infty$ is a sequence of non-negative terms. (This conclusion also comes from the fact that $(\mathbb{C}, d\nu_{\text{Gauss}})$ is a probability measure space.)

Next, by using the equality $S_{n+1} - S_n = 1 + \psi(n+1) - \log(n+1)$ of the previous argument and Lemma (3.1) (c) we have that

$$\lim_{n \rightarrow \infty} (S_{n+1} - S_n) = 1,$$

which proves that the sequence $\{S_n\}_{n=1}^\infty$ is unbounded and, moreover, implies that

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} = 1.$$

(Proof: $\lim_{n \rightarrow \infty} (S_{n+1} - S_n) = 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (S_{k+1} - S_k) = 1$
 $\Rightarrow \lim_{n \rightarrow \infty} \frac{S_n - S_0}{n} = 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{S_n}{n} = 1$.) This limit can also be proved directly by noting that

$$\begin{aligned} \frac{S_n}{n} &= \psi(n+1) - \frac{1}{n} \log n! \\ &= \psi(n+1) - \log n - \log \frac{(n!)^{\frac{1}{n}}}{n}, \end{aligned}$$

and thus, by using that $\psi(n+1) - \log n \rightarrow 0$ as $n \rightarrow \infty$ (Lemma 3.1 (a)) and that $\log \frac{(n!)^{\frac{1}{n}}}{n} \rightarrow e^{-1}$ as $n \rightarrow \infty$, we obtain the desired result $\lim_{n \rightarrow \infty} \frac{S_n}{n} = 1$.

In conclusion, we have proved the following.

Proposition 3.1 *The sequence $\{S_n\}_{n=0}^\infty$, where $S_n = S_{L^2(\mathbb{C}, d\nu_{\text{Gauss}})}(\xi_n)$ is the entropy of the n -th canonical basis element in $L^2(\mathbb{C}, d\nu_{\text{Gauss}})$ is an unbounded increasing sequence of non-negative terms, with the property $\lim_{n \rightarrow \infty} (S_{n+1} - S_n) = 1$ (which implies that $\lim_{n \rightarrow \infty} \frac{S_n}{n} = 1$).*

4 Entropies in \mathcal{B}_μ^2

As noted in Section 2, for calculating the entropies $S_n^\mu = S_{L^2(\mathbb{C} \times \mathbb{Z}_2, d\nu_\mu)}(\xi_n^\mu)$ of the elements of the canonical basis $\{\xi_n^\mu\}_{n=0}^\infty$ of the μ -deformed Segal-Bargmann

space \mathcal{B}_μ^2 , we need to consider the cases when n is even (in which case we have that $S_n^\mu = S_{L^2(\mathbb{C}, d\nu_{e,\mu})}(\xi_n)$) and when n is odd (in which case we have that $S_n^\mu = S_{L^2(\mathbb{C}, d\nu_{o,\mu})}(\xi_n)$). We begin by considering the even case. For $n = 0$ we have $\xi_0^\mu(z) = 1$ and then $S_0^\mu = 0$. So we are interested in calculating S_{2n}^μ for $n \geq 1$. Formula (2.15) tells us that

$$\begin{aligned} S_{2n}^\mu &= \int_{\mathbb{C}} |\xi_{2n}(z)|^2 \log |\xi_{2n}(z)|^2 d\nu_{e,\mu}(z) - \|\xi_{2n}\|_{L^2(\mathbb{C}, d\nu_{e,\mu})}^2 \log \|\xi_{2n}\|_{L^2(\mathbb{C}, d\nu_{e,\mu})}^2 \\ &= \frac{2^{\frac{1}{2}-\mu}}{\pi \Gamma(\mu + \frac{1}{2})} \int_{\mathbb{C}} \left| \frac{z^{2n}}{(\gamma_\mu(2n))^{\frac{1}{2}}} \right|^2 \log \left| \frac{z^{2n}}{(\gamma_\mu(2n))^{\frac{1}{2}}} \right|^2 K_{\mu-\frac{1}{2}}(|z|^2) |z|^{2\mu+1} dx dy. \end{aligned}$$

Since the log term in the integral of the right hand side is $\log |z^{2n}|^2 - \log \gamma_\mu(2n)$, we can write S_{2n}^μ as a difference of two integrals, $I_1 - I_2$ say, in which $I_2 = \log \gamma_\mu(2n) \|\xi_{2n}\|_{L^2(\mathbb{C}, d\nu_{e,\mu})}^2 = \log \gamma_\mu(2n)$. In I_1 we change (x, y) to polar coordinates (r, θ) , and then let $s = r^2$ to obtain

$$\begin{aligned} S_{2n}^\mu &= \frac{2^{\frac{1}{2}-\mu}}{\pi \Gamma(\mu + \frac{1}{2})} \int_{\mathbb{C}} \left| \frac{z^{2n}}{(\gamma_\mu(2n))^{\frac{1}{2}}} \right|^2 \log |z^{2n}|^2 K_{\mu-\frac{1}{2}}(|z|^2) |z|^{2\mu+1} dx dy \\ &\quad - \log \gamma_\mu(2n) \\ &= \frac{2^{\frac{1}{2}-\mu} 2}{\gamma_\mu(2n) \Gamma(\mu + \frac{1}{2})} \int_0^\infty r^{4n} (\log r^{4n}) K_{\mu-\frac{1}{2}}(r^2) r^{2\mu+2} dr - \log \gamma_\mu(2n) \\ &= \frac{2^{\frac{1}{2}-\mu}}{\gamma_\mu(2n) \Gamma(\mu + \frac{1}{2})} \int_0^\infty s^{2n} (\log s^{2n}) K_{\mu-\frac{1}{2}}(s) s^{\mu+\frac{1}{2}} ds - \log \gamma_\mu(2n). \end{aligned}$$

For calculating the integral $\int_0^\infty K_{\mu-\frac{1}{2}}(s) s^{\mu+2n+\frac{1}{2}} ds$, we define the function φ in a neighborhood of $\alpha = 1$ as

$$\varphi(\alpha) = \int_0^\infty s^{2n\alpha} K_{\mu-\frac{1}{2}}(s) s^{\mu+\frac{1}{2}} ds.$$

Observe that for $\mu > -\frac{1}{2}$, $n \in \mathbb{N}$ and α in a neighborhood of 1, one has that $2n\alpha + \mu + \frac{3}{2} > |\mu - \frac{1}{2}|$, so we can use formula (2.7) to write

$$\varphi(\alpha) = 2^{2n\alpha+\mu-\frac{1}{2}} \Gamma(n\alpha+1) \Gamma\left(\mu+n\alpha+\frac{1}{2}\right).$$

The derivative φ' is on the one hand

$$\varphi'(\alpha) = \int_0^\infty s^{2n\alpha} (\log s^{2n}) K_{\mu-\frac{1}{2}}(s) s^{\mu+\frac{1}{2}} ds,$$

and on the other hand

$$\begin{aligned}
\varphi'(\alpha) &= 2^{2n\alpha+\mu-\frac{1}{2}} \Gamma(n\alpha+1) n\Gamma' \left(\mu+n\alpha+\frac{1}{2} \right) \\
&\quad + 2^{2n\alpha+\mu-\frac{1}{2}} n\Gamma'(n\alpha+1) \Gamma \left(\mu+n\alpha+\frac{1}{2} \right) \\
&\quad + 2^{2n\alpha+\mu-\frac{1}{2}} 2n(\log 2) \Gamma(n\alpha+1) \Gamma \left(\mu+n\alpha+\frac{1}{2} \right) \\
&= 2^{2n\alpha+\mu-\frac{1}{2}} \Gamma(n\alpha+1) \Gamma \left(\mu+n\alpha+\frac{1}{2} \right) \begin{pmatrix} n\psi \left(\mu+n\alpha+\frac{1}{2} \right) \\ +n\psi(n\alpha+1) \\ +2n \log 2 \end{pmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
\varphi'(1) &= \int_0^\infty s^{2n} (\log s^{2n}) K_{\mu-\frac{1}{2}}(s) s^{\mu+\frac{1}{2}} ds \\
&= 2^{2n+\mu-\frac{1}{2}} \Gamma(n+1) \Gamma \left(\mu+n+\frac{1}{2} \right) \begin{pmatrix} n\psi \left(\mu+n+\frac{1}{2} \right) \\ +n\psi(n+1) \\ +2n \log 2 \end{pmatrix}.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
S_{2n}^\mu &= \frac{\Gamma(n+1) \Gamma \left(\mu+n+\frac{1}{2} \right) 2^{2n}}{\gamma_\mu(2n) \Gamma \left(\mu+\frac{1}{2} \right)} \left(n\psi \left(\mu+n+\frac{1}{2} \right) + n\psi(n+1) + \log 2^{2n} \right) \\
&\quad - \log \gamma_\mu(2n).
\end{aligned}$$

By using formula (2.1) for $\gamma_\mu(2n)$ we have that the entropy of the even elements ξ_{2n} is

$$S_{2n}^\mu = n \left(\psi \left(\mu+n+\frac{1}{2} \right) + \psi(n+1) \right) - \log \frac{\gamma_\mu(2n)}{2^{2n}}. \quad (4.1)$$

Note that this formula makes sense for $n=0$, obtaining the known result $S_0^\mu = 0$.

In the case $\mu=0$, formula (4.1) becomes

$$\begin{aligned}
S_{2n}^0 &= n \left(\psi \left(n+\frac{1}{2} \right) + \psi(n+1) \right) - \log \frac{(2n)!}{2^{2n}} \\
&= n \left(-\gamma - 2 \log 2 + 2 \sum_{k=1}^n \frac{1}{2k-1} - \gamma + \sum_{k=1}^n \frac{1}{k} \right) - \log (2n)! + 2n \log 2 \\
&= 2n \left(-\gamma + \sum_{k=1}^n \frac{1}{2k-1} + \frac{1}{2} \sum_{k=1}^n \frac{1}{k} \right) - \log (2n)! \\
&= 2n \left(-\gamma + \sum_{k=1}^{2n} \frac{1}{k} \right) - \log (2n)!,
\end{aligned}$$

which is (1.1) for even positive integers, as expected.

Since $(\mathbb{C}, d\nu_{e,\mu})$ is a probability measure space, we have that $S_{2n}^\mu \geq 0$ for all $n = 0, 1, 2, \dots$. But we can arrive at this conclusion directly from the formula obtained for S_{2n}^μ as follows. Observe that for $n \in \mathbb{N}$ we can write formula (2.1) as

$$\frac{\gamma_\mu(2n)}{2^{2n}} = n! \prod_{k=1}^n \left(\mu + k - \frac{1}{2} \right). \quad (4.2)$$

Then

$$\begin{aligned} S_{2n}^\mu &= n\psi\left(\mu + n + \frac{1}{2}\right) + n\psi(n+1) - \log\left(n! \prod_{k=1}^n \left(\mu + k - \frac{1}{2}\right)\right) \\ &= \sum_{k=1}^n \left(\psi\left(\mu + n + \frac{1}{2}\right) - \log\left(\mu + k - \frac{1}{2}\right) + \psi(n+1) - \log(k) \right). \end{aligned}$$

Lemma 3.1(a) gives us that $\psi\left(\mu + n + \frac{1}{2}\right) - \log\left(\mu + k - \frac{1}{2}\right) > 0$ and that $\psi(n+1) - \log(k) > 0$ for all $k = 1, \dots, n$. So we conclude that $S_{2n}^\mu > 0$, as wanted. Moreover, observe that for fixed $n \in \mathbb{N}$, we have that (again by Lemma 3.1(a)) $\psi\left(\mu + n + \frac{1}{2}\right) - \log\left(\mu + k - \frac{1}{2}\right) \rightarrow 0$ as $\mu \rightarrow +\infty$, and so

$$\lim_{\mu \rightarrow +\infty} S_{2n}^\mu = \sum_{k=1}^n (\psi(n+1) - \log(k)) = n\psi(n+1) - \log n!.$$

That is, for $n \in \mathbb{N}$ fixed we have that

$$\lim_{\mu \rightarrow +\infty} S_{2n}^\mu = S_n.$$

Let us consider the particular case when $\mu = \frac{1}{2} + m$, $m = 0, 1, 2, \dots$. Formula (4.2) becomes in this case

$$\frac{\gamma_{\frac{1}{2}+m}(2n)}{2^{2n}} = n! \prod_{k=1}^n (k+m) = \frac{n!(m+n)!}{m!},$$

and then formula (4.1) gives us

$$\begin{aligned} S_{2n}^{\frac{1}{2}+m} &= n(\psi(n+m+1) + \psi(n+1)) - \log \frac{n!(m+n)!}{m!} \\ &= (n+m)\psi(n+m+1) - \log(m+n)! + n\psi(n+1) \\ &\quad - \log n! - m\psi(n+m+1) + \log m! \\ &= S_{n+m} + S_n - m \left(\sum_{k=0}^{n-1} \frac{1}{m+k+1} + \psi(m+1) \right) + \log m! \\ &= S_{n+m} + S_n - S_m - \sum_{k=1}^n \frac{m}{m+k}. \end{aligned}$$

That is, for $n, m = 0, 1, 2, \dots$, we have the formula

$$S_{n+m} + S_n - S_m = S_{2n}^{\frac{1}{2}+m} + \sum_{k=1}^n \frac{m}{m+k},$$

which shows that the values of the entropies S_{n+m} , S_n and S_m (of the undeformed case) are related by means of the entropy $S_{2n}^{\frac{1}{2}+m}$ corresponding to the $(m + \frac{1}{2})$ -deformed case.

We claim that $\{S_{2n}^\mu\}_{n=0}^\infty$ is an increasing sequence for fixed $\mu > -\frac{1}{2}$. In fact, we have that

$$\begin{aligned} S_{2n+2}^\mu &= (n+1) \psi\left(\mu + n + \frac{3}{2}\right) + (n+1) \psi(n+2) - \log \frac{\gamma_\mu(2n+2)}{2^{2n+2}} \\ &= n \left(\frac{1}{\mu + n + \frac{1}{2}} + \psi\left(\mu + n + \frac{1}{2}\right) \right) + \psi\left(\mu + n + \frac{3}{2}\right) \\ &\quad + n \left(\frac{1}{n+1} + \psi(n+1) \right) + \psi(n+2) \\ &\quad - \log \frac{(2n+2)(2n+1+2\mu)\gamma_\mu(2n)}{2^2 2^{2n}} \\ &= S_{2n}^\mu + \psi\left(\mu + n + \frac{3}{2}\right) - \log\left(\mu + n + \frac{1}{2}\right) \\ &\quad + \psi(n+2) - \log(n+1) + \frac{n}{\mu + n + \frac{1}{2}} + \frac{n}{n+1}. \end{aligned}$$

Lemma 3.1(a) gives us $\psi\left(\mu + n + \frac{3}{2}\right) - \log\left(\mu + n + \frac{1}{2}\right) > 0$ and $\psi(n+2) - \log(n+1) > 0$. Thus we have that $S_{2n+2}^\mu - S_{2n}^\mu > 0$, as wanted. Lemma 3.1(a) also tells us that $\psi\left(\mu + n + \frac{3}{2}\right) - \log\left(\mu + n + \frac{1}{2}\right) \rightarrow 0$ and $\psi(n+2) - \log(n+1) \rightarrow 0$ as $n \rightarrow \infty$. Thus, for fixed $\mu > -\frac{1}{2}$, we have by the expression above that $\lim_{n \rightarrow \infty} (S_{2n+2}^\mu - S_{2n}^\mu) = 2$. In particular we see that the sequence $\{S_{2n}^\mu\}_{n=0}^\infty$ is unbounded. This limit implies that $\lim_{n \rightarrow \infty} \frac{S_{2n}}{2n} = 1$, but we can give a direct proof of this last result by noting that

$$\begin{aligned} \frac{S_{2n}^\mu}{2n} &= \frac{1}{2} \left(\psi\left(\mu + n + \frac{1}{2}\right) + \psi(n+1) \right) - \frac{1}{2n} \log \frac{\gamma_\mu(2n)}{2^{2n}} \\ &= \frac{1}{2} \left(\psi\left(\mu + n + \frac{1}{2}\right) + \psi(n+1) \right) - \log \frac{(\gamma_\mu(2n))^{\frac{1}{2n}}}{2n} - \log n \\ &= \frac{1}{2} \left(\psi\left(\mu + n + \frac{1}{2}\right) - \log n + \psi(n+1) - \log n \right) - \log \frac{(\gamma_\mu(2n))^{\frac{1}{2n}}}{2n}. \end{aligned}$$

Lemma 3.1(b) tells us that $\psi\left(\mu + n + \frac{1}{2}\right) - \log n \rightarrow 0$ and $\psi(n+1) - \log n \rightarrow 0$ as $n \rightarrow \infty$. Lemma 3.2 tells us that $\log \frac{(\gamma_\mu(2n))^{\frac{1}{2n}}}{2n} \rightarrow -1$ as $n \rightarrow \infty$. Then we have that $\lim_{n \rightarrow \infty} \frac{S_{2n}}{2n} \rightarrow 1$, as wanted.

In conclusion, we have proved the following theorem.

Theorem 4.1 *The entropy of ξ_{2n}^μ is given by*

$$S_{2n}^\mu = n \left(\psi \left(\mu + n + \frac{1}{2} \right) + \psi(n+1) \right) - \log \frac{\gamma_\mu(2n)}{2^{2n}},$$

where $\mu > -\frac{1}{2}$ and $n = 0, 1, \dots$. For fixed $\mu > -\frac{1}{2}$, the sequence $\{S_{2n}^\mu\}_{n=1}^\infty$ is an unbounded increasing sequence of positive terms such that

$$\lim_{n \rightarrow \infty} (S_{2n+2}^\mu - S_{2n}^\mu) = 2,$$

which implies that

$$\lim_{n \rightarrow \infty} \frac{S_{2n}^\mu}{2n} = 1.$$

For fixed $n \in \mathbb{N}$, we have that

$$\lim_{\mu \rightarrow +\infty} S_{2n}^\mu = S_n,$$

where $S_n = S_n^0$.

For $n, m = 0, 1, 2, \dots$, we have that

$$S_{n+m} + S_n - S_m = S_{2n}^{\frac{1}{2}+m} + \sum_{k=0}^{n-1} \frac{m}{m+k+1}.$$

We now calculate the entropies of the odd functions ξ_{2n+1} , $n = 0, 1, 2, \dots$. The steps we will follow in the calculations are analogues of the even case. Since $S_{2n+1}^\mu = S_{L^2(\mathbb{C}, d\nu_{o,\mu})}(\xi_{2n+1})$ we have that

$$\begin{aligned} S_{2n+1}^\mu &= \frac{2^{\frac{1}{2}-\mu}}{\pi \Gamma(\mu + \frac{1}{2})} \\ &\cdot \int_{\mathbb{C}} \left| \frac{z^{2n+1}}{(\gamma_\mu(2n+1))^{\frac{1}{2}}} \right|^2 \log \left| \frac{z^{2n+1}}{(\gamma_\mu(2n+1))^{\frac{1}{2}}} \right|^2 K_{\mu+\frac{1}{2}}(|z|^2) |z|^{2\mu+1} dx dy \\ &= \frac{2^{\frac{1}{2}-\mu} 2}{\gamma_\mu(2n+1) \Gamma(\mu + \frac{1}{2})} \int_0^\infty r^{4n+2} (\log r^{4n+2}) K_{\mu+\frac{1}{2}}(r^2) r^{2\mu+2} dr \\ &\quad - \log \gamma_\mu(2n+1) \\ &= \frac{2^{\frac{1}{2}-\mu}}{\gamma_\mu(2n+1) \Gamma(\mu + \frac{1}{2})} \int_0^\infty s^{2n+1} (\log s^{2n+1}) K_{\mu+\frac{1}{2}}(s) s^{\mu+\frac{1}{2}} ds \\ &\quad - \log \gamma_\mu(2n+1). \end{aligned}$$

We define

$$\phi(\alpha) = \int_0^\infty s^{(2n+1)\alpha} K_{\mu+\frac{1}{2}}(s) s^{\mu+\frac{1}{2}} ds.$$

Since for $\mu > -\frac{1}{2}$, $n \in \mathbb{N} \cup \{0\}$ and α in a neighborhood of 1, one has that $(2n+1)\alpha + \mu + \frac{3}{2} > \left|\mu + \frac{1}{2}\right|$, we can use formula (2.7) to write

$$\phi(\alpha) = 2^{(2n+1)\alpha + \mu - \frac{1}{2}} \Gamma\left(\left(n + \frac{1}{2}\right)\alpha + \frac{1}{2}\right) \Gamma\left(\left(n + \frac{1}{2}\right)\alpha + \mu + 1\right).$$

By calculating the derivative $\phi'(1)$ in two different ways as we did in the even case, we get

$$\begin{aligned} \phi'(1) &= \int_0^\infty s^{2n+1} \log s^{2n+1} K_{\mu+\frac{1}{2}}(s) s^{\mu+\frac{1}{2}} ds \\ &= 2^{2n+\mu+\frac{1}{2}} \Gamma(n+1) \Gamma\left(\mu + n + \frac{3}{2}\right) \begin{pmatrix} \frac{2n+1}{2} \psi\left(\mu + n + \frac{3}{2}\right) \\ + \frac{2n+1}{2} \psi(n+1) \\ + (2n+1) \log 2 \end{pmatrix}. \end{aligned}$$

Thus, by using formula (2.2) for $\gamma_\mu(2n+1)$ we find that the entropy of ξ_{2n+1} is

$$S_{2n+1}^\mu = \left(n + \frac{1}{2}\right) \left(\psi\left(\mu + n + \frac{3}{2}\right) + \psi(n+1)\right) - \log \frac{\gamma_\mu(2n+1)}{2^{2n+1}}. \quad (4.3)$$

In the case $\mu = 0$ this formula becomes

$$\begin{aligned} S_{2n+1}^0 &= \left(n + \frac{1}{2}\right) \left(\psi\left(n + \frac{3}{2}\right) + \psi(n+1)\right) - \log \frac{(2n+1)!}{2^{2n+1}} \\ &= \left(n + \frac{1}{2}\right) \left(\frac{1}{n + \frac{1}{2}} + \psi\left(n + \frac{1}{2}\right) + \psi(n+1)\right) \\ &\quad - \log(2n+1)! + (2n+1) \log 2 \\ &= (2n+1) \left(-\gamma + \frac{1}{2n+1} + \sum_{k=1}^n \frac{1}{2k-1} + \frac{1}{2} \sum_{k=1}^n \frac{1}{k}\right) - \log(2n+1)! \\ &= (2n+1) \left(-\gamma + \sum_{k=1}^{2n+1} \frac{1}{k}\right) - \log(2n+1)!, \end{aligned}$$

which is (1.1) for odd positive integers.

Observe that for $n \in \mathbb{N}$ we can write formula (2.2) as

$$\frac{\gamma_\mu(2n+1)}{2^{2n+1}} = n! \prod_{k=1}^{n+1} \left(\mu + k - \frac{1}{2}\right).$$

Thus (4.3) can be written as

$$\begin{aligned}
S_{2n+1}^\mu &= \sum_{k=1}^n \left(\psi \left(\mu + n + \frac{3}{2} \right) - \log \left(\mu + k - \frac{1}{2} \right) \right) \\
&\quad + \frac{1}{2} \left(\psi \left(\mu + n + \frac{3}{2} \right) - \log \left(\mu + n + \frac{1}{2} \right) \right) \\
&\quad + \left(n + \frac{1}{2} \right) \psi(n+1) - \log n! \\
&\quad - \frac{1}{2} \log \left(\mu + n + \frac{1}{2} \right).
\end{aligned} \tag{4.4}$$

For fixed $n = 0, 1, 2, \dots$, we have by Lemma 3.1(a) that $\psi(\mu + n + \frac{3}{2}) - \log(\mu + k - \frac{1}{2}) \rightarrow 0$ and $\psi(\mu + n + \frac{3}{2}) - \log(\mu + n + \frac{1}{2}) \rightarrow 0$ as $\mu \rightarrow +\infty$. Thus, because of the last term of the right hand side in (4.4), we have that $\lim_{\mu \rightarrow +\infty} S_{2n+1}^\mu = -\infty$. That is, negative entropies do occur in the odd case. (Recall that $(\mathbb{C}, d\nu_{o,\mu})$ is *not* a probability measure space for $\mu \neq 0$.) Nevertheless we will see now that for fixed $\mu > -\frac{1}{2}$ the sequence $\{S_{2n+1}^\mu\}_{n=0}^\infty$ is increasing and unbounded, and so it is eventually positive. We have that

$$\begin{aligned}
S_{2n+3}^\mu &= \left(n + \frac{1}{2} \right) \left(\psi \left(\mu + n + \frac{3}{2} \right) + \psi(n+1) \right) - \log \frac{\gamma_\mu(2n+1)}{2^{2n+1}} \\
&\quad + \left(n + \frac{3}{2} \right) \left(\frac{1}{\mu + n + \frac{3}{2}} + \frac{1}{n+1} \right) + \psi \left(\mu + n + \frac{3}{2} \right) \\
&\quad + \psi(n+1) - \log \left((n+1) \left(\mu + n + \frac{3}{2} \right) \right) \\
&= S_{2n+1}^\mu + \left(n + \frac{3}{2} \right) \left(\frac{1}{\mu + n + \frac{3}{2}} + \frac{1}{n+1} \right) \\
&\quad + \psi \left(\mu + n + \frac{3}{2} \right) - \log \left(\mu + n + \frac{3}{2} \right) + \psi(n+1) - \log(n+1) \\
&> S_{2n+1}^\mu + \left(n + \frac{3}{2} \right) \left(\frac{1}{\mu + n + \frac{3}{2}} + \frac{1}{n+1} \right) - \frac{1}{\mu + n + \frac{3}{2}} - \frac{1}{n+1} \\
&= S_{2n+1}^\mu + \left(n + \frac{1}{2} \right) \left(\frac{1}{\mu + n + \frac{3}{2}} + \frac{1}{n+1} \right) \\
&> S_{2n+1}^\mu,
\end{aligned}$$

where we used Lemma 3.1(c). This proves that the sequence $\{S_{2n+1}^\mu\}_{n=0}^\infty$ is increasing. Moreover, since

$$\begin{aligned}
S_{2n+3}^\mu - S_{2n+1}^\mu &= \left(n + \frac{3}{2} \right) \left(\frac{1}{\mu + n + \frac{3}{2}} + \frac{1}{n+1} \right) \\
&\quad + \psi \left(\mu + n + \frac{3}{2} \right) - \log \left(\mu + n + \frac{3}{2} \right) \\
&\quad + \psi(n+1) - \log(n+1)
\end{aligned}$$

and by Lemma 3.1(c) we have that $\psi\left(\mu + n + \frac{3}{2}\right) - \log\left(\mu + n + \frac{3}{2}\right) \rightarrow 0$ and also that $\psi(n+1) - \log(n+1) \rightarrow 0$ as $n \rightarrow \infty$, then we conclude that $\lim_{n \rightarrow \infty} (S_{2n+3}^\mu - S_{2n+1}^\mu) = 2$, which implies the unboundedness of the sequence $\{S_{2n+1}^\mu\}_{n=0}^\infty$. This limit implies that $\lim_{n \rightarrow \infty} \frac{S_{2n+1}^\mu}{2n+1} = 1$, but a direct proof of this is as follows. Note that

$$\begin{aligned} \frac{S_{2n+1}^\mu}{2n+1} &= \frac{1}{2} \left(\psi\left(\mu + n + \frac{3}{2}\right) + \psi(n+1) \right) - \log \frac{(\gamma_\mu(2n+1))^{\frac{1}{2n+1}}}{2n+1} \\ &\quad - \log\left(n + \frac{1}{2}\right) \\ &= \frac{1}{2} \left(\psi\left(\mu + n + \frac{3}{2}\right) - \log\left(n + \frac{1}{2}\right) + \psi(n+1) - \log\left(n + \frac{1}{2}\right) \right) \\ &\quad - \log \frac{(\gamma_\mu(2n+1))^{\frac{1}{2n+1}}}{2n+1}. \end{aligned}$$

Note that Lemmas 3.1(b) and 3.2 give us that $\psi\left(\mu + n + \frac{3}{2}\right) - \log\left(n + \frac{1}{2}\right) \rightarrow 0$, $\psi(n+1) - \log\left(n + \frac{1}{2}\right) \rightarrow 0$ and $\log \frac{(\gamma_\mu(2n+1))^{\frac{1}{2n+1}}}{2n+1} \rightarrow -1$ as $n \rightarrow \infty$. Then we have that $\frac{S_{2n+1}^\mu}{2n+1} \rightarrow 1$ as $n \rightarrow \infty$.

Thus, we have proved the following theorem.

Theorem 4.2 *The entropy of ξ_{2n+1}^μ is given by*

$$S_{2n+1}^\mu = \left(n + \frac{1}{2}\right) \left(\psi\left(\mu + n + \frac{3}{2}\right) + \psi(n+1) \right) - \log \frac{\gamma_\mu(2n+1)}{2^{2n+1}},$$

where $\mu > -\frac{1}{2}$ and $n = 0, 1, 2, \dots$. For fixed $\mu > -\frac{1}{2}$, the sequence $\{S_{2n+1}^\mu\}_{n=0}^\infty$ is an unbounded increasing sequence such that

$$\lim_{n \rightarrow \infty} (S_{2n+3}^\mu - S_{2n+1}^\mu) = 2,$$

which implies that

$$\lim_{n \rightarrow \infty} \frac{S_{2n+1}^\mu}{2n+1} = 1.$$

For fixed $n = 0, 1, 2, \dots$, we have that

$$\lim_{\mu \rightarrow +\infty} S_{2n+1}^\mu = -\infty.$$

We can relate the entropies S_{2n+1}^μ with the entropies S_{2n}^μ as follows. We

note that

$$\begin{aligned}
S_{2n+1}^\mu &= \frac{2n+1}{2} \left(\psi \left(\mu + n + \frac{3}{2} \right) + \psi(n+1) \right) - \log \frac{\gamma_\mu(2n+1)}{2^{2n+1}} \\
&= \left(n + \frac{1}{2} \right) \left(\frac{1}{\mu + n + \frac{1}{2}} + \psi \left(\mu + n + \frac{1}{2} \right) + \psi(n+1) \right) \\
&\quad - \log \frac{\gamma_\mu(2n+1)}{2^{2n+1}} \\
&= \left(1 + \frac{1}{2n} \right) \left(S_{2n}^\mu + \log \frac{\gamma_\mu(2n)}{2^{2n}} \right) + \frac{n + \frac{1}{2}}{\mu + n + \frac{1}{2}} - \log \frac{\gamma_\mu(2n+1)}{2^{2n+1}} \\
&= \left(1 + \frac{1}{2n} \right) S_{2n}^\mu + \frac{n + \frac{1}{2}}{\mu + n + \frac{1}{2}} + \log \frac{(\gamma_\mu(2n))^{\frac{1}{2n}}}{2 \left(\mu + n + \frac{1}{2} \right)}.
\end{aligned}$$

So we have that

$$S_{2n+1}^\mu - S_{2n}^\mu = \frac{S_{2n}^\mu}{2n} + \frac{n + \frac{1}{2}}{\mu + n + \frac{1}{2}} + \log \frac{(\gamma_\mu(2n))^{\frac{1}{2n}}}{2 \left(\mu + n + \frac{1}{2} \right)}.$$

By using Lemma 3.2 and Theorem 4.1 we obtain

$$\lim_{n \rightarrow \infty} (S_{2n+1}^\mu - S_{2n}^\mu) = 1.$$

Similarly one has that

$$S_{2n}^\mu - S_{2n-1}^\mu = \frac{S_{2n-1}^\mu}{2n-1} + 1 + \log \frac{\gamma_\mu(2n-1)^{\frac{1}{2n-1}}}{2n}.$$

Lemma 3.2 and Theorem 4.2 allow us to conclude

$$\lim_{n \rightarrow \infty} (S_{2n}^\mu - S_{2n-1}^\mu) = 1.$$

Finally, observe that we can express the formulas (4.1) and (4.3) in terms of the characteristic function θ of the odd positive integers as

$$S_n^\mu = \frac{n}{2} \left(\psi \left(\mu + \frac{n + \theta(n) + 1}{2} \right) + \psi \left(\frac{n + \theta(n+1) + 1}{2} \right) \right) - \log \frac{\gamma_\mu(n)}{2^n}.$$

From this formula one can obtain at once the case $\mu = 0$ (formula (1.1)) by using the identity

$$\psi \left(\frac{n+1}{2} \right) + \psi \left(\frac{n+2}{2} \right) = 2\psi(n+1) - 2\log 2,$$

whose proof is an easy exercise by induction.

Combining Theorems 4.1 and 4.2 with the previous results, we have the following.

Theorem 4.3 *Let $\mu > -\frac{1}{2}$ be fixed. The entropy S_n^μ is given by*

$$S_n^\mu = \frac{n}{2} \left(\psi \left(\mu + \frac{n + \theta(n) + 1}{2} \right) + \psi \left(\frac{n + \theta(n+1) + 1}{2} \right) \right) - \log \frac{\gamma_\mu(n)}{2^n}.$$

The sequence $\{S_n^\mu\}_{n=0}^\infty$ of entropies is such that the subsequences of even terms $\{S_{2n}^\mu\}_{n=1}^\infty$ and of odd terms $\{S_{2n+1}^\mu\}_{n=0}^\infty$ are increasing, the former being positive and the latter being eventually positive. Moreover, we have that

$$\lim_{n \rightarrow \infty} (S_{n+1}^\mu - S_n^\mu) = 1,$$

which shows that the sequence $\{S_n^\mu\}_{n=0}^\infty$ is unbounded and implies that

$$\lim_{n \rightarrow \infty} \frac{S_n^\mu}{n} = 1.$$

5 Entropies in $L^2(\mathbb{R}, dg_\mu)$

Following the same sort of ideas we used in the previous section, we will calculate in this section the entropies of monomials $t^n \in L^2(\mathbb{R}, dg_\mu)$, $n = 1, 2, \dots$ (In the case $n = 0$ we obtain from the definition that $S_{L^2(\mathbb{R}, dg_\mu)}(1) = 0$.) That is, for $n = 1, 2, \dots$ we will calculate explicitly

$$\begin{aligned} & S_{L^2(\mathbb{R}, dg_\mu)}(t^n) \\ &= \frac{1}{\Gamma(\mu + \frac{1}{2})} \int_{\mathbb{R}} |t^n|^2 \log |t^n|^2 \exp(-t^2) |t|^{2\mu} dt - \|t^n\|_{L^2(\mathbb{R}, dg_\mu)}^2 \log \|t^n\|_{L^2(\mathbb{R}, dg_\mu)}^2 \\ &= \frac{1}{\Gamma(\mu + \frac{1}{2})} \int_0^\infty u^n (\log u^n) \exp(-u) u^{\mu - \frac{1}{2}} du - \|t^n\|_{L^2(\mathbb{R}, dg_\mu)}^2 \log \|t^n\|_{L^2(\mathbb{R}, dg_\mu)}^2. \end{aligned}$$

A direct calculation gives us

$$\|t^n\|_{L^2(\mathbb{R}, dg_\mu)}^2 = \frac{\Gamma(n + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})}.$$

Next, define the function

$$\eta(\alpha) = \int_0^\infty u^{n\alpha} \exp(-u) u^{\mu - \frac{1}{2}} du = \Gamma\left(n\alpha + \mu + \frac{1}{2}\right)$$

in a neighborhood of $\alpha = 1$. By calculating the derivative $\eta'(1)$ in two different ways (as we did in previous section) we find that

$$\int_0^\infty u^n (\log u^n) \exp(-u) u^{\mu - \frac{1}{2}} du = n\psi\left(n + \mu + \frac{1}{2}\right) \Gamma\left(n + \mu + \frac{1}{2}\right).$$

Then the entropy $S_{L^2(\mathbb{R}, dg_\mu)}(t^n)$ is

$$S_{L^2(\mathbb{R}, dg_\mu)}(t^n) = \frac{\Gamma(n + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \left(n\psi\left(n + \mu + \frac{1}{2}\right) - \log \frac{\Gamma(n + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \right). \quad (5.1)$$

When $n = 1$ formula (5.1) becomes

$$\begin{aligned} S_{L^2(\mathbb{R}, dg_\mu)}(t) &= \frac{\Gamma(\mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})} \left(\psi\left(\mu + \frac{3}{2}\right) - \log \frac{\Gamma(\mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})} \right) \\ &= \left(\mu + \frac{1}{2}\right) \left(\psi\left(\mu + \frac{3}{2}\right) - \log\left(\mu + \frac{1}{2}\right) \right). \end{aligned}$$

By using that $S_{L^2(\mathbb{R}, dg_\mu)}(t)$ is homogeneous of degree 2 we can calculate the entropy of the monomial $\zeta_1^\mu(t) = \left(\frac{2}{1+2\mu}\right)^{\frac{1}{2}} t$, which is the second element of the canonical basis $\{\zeta_n^\mu\}_{n=0}^\infty$ of $L^2(\mathbb{R}, dg_\mu)$. In fact, we have that $S_{L^2(\mathbb{R}, dg_\mu)}(\zeta_1^\mu) = \frac{2}{1+2\mu} S_{L^2(\mathbb{R}, dg_\mu)}(t)$, and then

$$S_{L^2(\mathbb{R}, dg_\mu)}(\zeta_1^\mu) = \psi\left(\mu + \frac{3}{2}\right) - \log\left(\mu + \frac{1}{2}\right). \quad (5.2)$$

When $\mu = 0$ this formula becomes

$$S_{L^2(\mathbb{R}, dg)}(\zeta_1^0) = \psi\left(\frac{3}{2}\right) - \log\left(\frac{1}{2}\right) = 2 - \log 2 - \gamma,$$

using $\psi\left(\frac{3}{2}\right) = 2 - 2\log 2 - \gamma$, which is (1.2) as expected.

Unfortunately we can not continue the previous procedure in order to obtain explicit formulas for the entropies of $\zeta_n^\mu \in L^2(\mathbb{R}, dg_\mu)$ with $n \geq 2$, since for those values of n the polynomials ζ_n^μ are not longer monomials, and then (5.1) is not useful. Nevertheless we will study some properties of the sequence $\{s_n^\mu\}_{n=0}^\infty$, where $s_n^\mu := S_{L^2(\mathbb{R}, dg_\mu)}(t^n)$, and compare them with the results obtained in Section 4.

Before that, recall that the μ -deformed Segal-Bargmann transform $B_\mu : L^2(\mathbb{R}, dg_\mu) \rightarrow \mathcal{B}_\mu^2$ is such that $B_\mu(\zeta_n^\mu) = \xi_n^\mu$, $n = 0, 1, \dots$. When $n = 0$ we have $\zeta_0^\mu(t) = 1$, $\xi_0^\mu(z) = 1$, and $S_{L^2(\mathbb{R}, dg_\mu)}(1) = S_{L^2(\mathbb{C}, d\nu_{e,\mu})}(1) = 0$. So in this case we see that B_μ preserves entropy. Let us consider the case $n = 1$. Formula (4.3) gives us

$$S_{L^2(\mathbb{C}, d\nu_{o,\mu})}(B_\mu(\zeta_1^\mu)) = \frac{1}{2} \left(\psi\left(\mu + \frac{3}{2}\right) + \psi(1) \right) - \log\left(\mu + \frac{1}{2}\right).$$

This formula, $\psi(1) = -\gamma$, and (5.2) give us that

$$S_{L^2(\mathbb{C}, d\nu_{o,\mu})}(B_\mu(\zeta_1^\mu)) - S_{L^2(\mathbb{R}, dg_\mu)}(\zeta_1^\mu) = -\frac{1}{2} \left(\psi\left(\mu + \frac{3}{2}\right) + \gamma \right). \quad (5.3)$$

Observe that $\lim_{\mu \rightarrow -\frac{1}{2}+} (\psi(\mu + \frac{3}{2}) + \gamma) = \psi(1) + \gamma = 0$, and that $\mu \mapsto \psi(\mu + \frac{3}{2}) + \gamma$ is an increasing function since we have that $\psi'(x) > 0$ for $x > 0$ (see [M-O-S], p. 14). Then $\psi(\mu + \frac{3}{2}) + \gamma > 0$ for $\mu > -\frac{1}{2}$, and thus formula (5.3) tells us that $S_{L^2(\mathbb{C}, d\nu_{o,\mu})}(B_\mu(\zeta_1^\mu)) < S_{L^2(\mathbb{R}, dg_\mu)}(\zeta_1^\mu)$. That is, the μ -deformed Segal-Bargmann transform B_μ decreases the entropy of ζ_1^μ . We have already noted that B_μ preserves the entropy of ζ_0^μ . It seems reasonable to conjecture that B_μ increases the entropy of other functions in $L^2(\mathbb{R}, dg_\mu)$. (This is known to be true in the case $\mu = 0$. See [Snt1].)

As happens in the case of the sequence of entropies $\{S_n^\mu\}_{n=0}^\infty$ in the previous section, the sequence of entropies $\{s_n^\mu\}_{n=0}^\infty$ is unbounded as we will prove now. By using the asymptotics

$$\psi(z) = \log z + O(z^{-1}), \quad (5.4)$$

valid for $|\arg z| < \pi$ and $z \rightarrow \infty$ (see [M-O-S], p. 18), and Stirling's formula

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + O(1), \quad (5.5)$$

also valid for $|\arg z| < \pi$ and $z \rightarrow \infty$ (see [M-O-S], p. 12), we see that for large n the term in parentheses in the right hand side of (5.1) behaves like

$$\begin{aligned} & n \left(\log \left(n + \mu + \frac{1}{2} \right) + O(n^{-1}) \right) - (n + \mu) \log \left(n + \mu + \frac{1}{2} \right) \\ & + \left(n + \mu + \frac{1}{2} \right) + O(1) \\ & = n - \mu \log \left(n + \mu + \frac{1}{2} \right) + O(1), \end{aligned}$$

which is unbounded. In turn this implies that the sequence of entropies $\{s_n^\mu\}_{n=0}^\infty$ is unbounded, as wanted.

Now let us see that the sequence $\{s_n^\mu\}_{n=0}^\infty$ is increasing (as the sequence $\{S_n^\mu\}_{n=0}^\infty$ is). First note that Lemma (3.1) (a) gives us

$$s_1^\mu = \left(\mu + \frac{1}{2} \right) \left(\psi \left(\mu + \frac{3}{2} \right) - \log \left(\mu + \frac{1}{2} \right) \right) > 0 = s_0^\mu,$$

so let us prove that $s_{n+1}^\mu > s_n^\mu$ for $n \geq 1$. Observe that

$$\frac{\Gamma(n + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} = \prod_{k=1}^n \left(k + \mu - \frac{1}{2} \right).$$

So we can write (5.1) as

$$\begin{aligned} s_n^\mu &= \frac{\Gamma(n + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \left(n \psi \left(n + \mu + \frac{1}{2} \right) - \log \prod_{k=1}^n \left(k + \mu - \frac{1}{2} \right) \right) \\ &= \frac{\Gamma(n + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \sum_{k=1}^n \left(\psi \left(n + \mu + \frac{1}{2} \right) - \log \left(k + \mu - \frac{1}{2} \right) \right). \end{aligned}$$

Then, by using Lemma (3.1) (a) we get

$$\begin{aligned}
s_{n+1}^\mu &= \frac{\Gamma(n + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})} \sum_{k=1}^{n+1} \left(\psi\left(n + \mu + \frac{3}{2}\right) - \log\left(k + \mu - \frac{1}{2}\right) \right) \\
&= \frac{\Gamma(n + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})} \sum_{k=1}^n \left(\frac{1}{n + \mu + \frac{1}{2}} + \psi\left(n + \mu + \frac{1}{2}\right) - \log\left(k + \mu - \frac{1}{2}\right) \right) \\
&\quad + \frac{\Gamma(n + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})} \left(\psi\left(n + \mu + \frac{3}{2}\right) - \log\left(n + \mu + \frac{1}{2}\right) \right) \\
&= \frac{n\Gamma(n + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} + \left(n + \mu + \frac{1}{2}\right) s_n^\mu \\
&\quad + \frac{\Gamma(n + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})} \left(\psi\left(n + \mu + \frac{3}{2}\right) - \log\left(n + \mu + \frac{1}{2}\right) \right) \\
&> s_n^\mu,
\end{aligned}$$

which proves that $\{s_n^\mu\}_{n=0}^\infty$ is increasing, as wanted. In particular we have that the sequence $\{s_n^\mu\}_{n=1}^\infty$ is positive. (Recall that since $L^2(\mathbb{R}, dg_\mu)$ is a probability measure space, we have that $s_n^\mu \geq 0$ for all $n = 0, 1, \dots$)

Finally, observe that from formulas (2.1), (3.2) and (4.1) we see that (5.1) can be written as

$$s_n^\mu = \frac{\Gamma(n + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} (S_{2n}^\mu - S_n).$$

We know that $\lim_{n \rightarrow \infty} \frac{S_{2n}^\mu}{2n} = 1$ (Theorem 4.1) and $\lim_{n \rightarrow \infty} \frac{S_n}{n} = 1$ (Proposition 3.1). Then we have that

$$\lim_{n \rightarrow +\infty} \frac{s_n^\mu}{n\Gamma(n + \mu + \frac{1}{2})} = \left(\Gamma\left(\mu + \frac{1}{2}\right) \right)^{-1}.$$

So the sequence $\{s_n^\mu\}_{n=0}^\infty$ diverges to infinity much faster than the sequence $\{S_n^\mu\}_{n=0}^\infty$ does.

6 μ -deformed Energies

In this section we study two entropy-energy inequalities, known as *reverse log-Sobolev inequalities* (in the μ -deformed Segal-Bargmann space \mathcal{B}_μ^2) that are proved in [A-S.1]. We first quote the appropriate definition of energy from [A-S.1] and then calculate it for the functions in the canonical basis of \mathcal{B}_μ^2 . Since we already have calculated the entropies for these functions, we can then proceed to the analysis of the two reverse log-Sobolev inequalities.

Definition 6.1 For $f \in \mathcal{B}_{e,\mu}^2$ we define its μ -deformed energy $E_{e,\mu}(f)$ as

$$E_{e,\mu}(f) = \int_{\mathbb{C}} |f(z)|^2 |z|^2 d\nu_{e,\mu}(z).$$

For $f \in \mathcal{B}_{o,\mu}^2$ we define its μ -deformed energy $E_{o,\mu}(f)$ as

$$E_{o,\mu}(f) = \int_{\mathbb{C}} |f(z)|^2 |z|^2 d\nu_{o,\mu}(z).$$

In general, for $f \in \mathcal{B}_{\mu}^2$ we define its μ -deformed energy $E_{\mu}(f)$ as $E_{\mu}(f) = E_{e,\mu}(f_e) + E_{o,\mu}(f_o)$.

We will denote by E_n^{μ} to the μ -deformed energy $E_{\mu}(\xi_n^{\mu})$, so we have $E_{2n}^{\mu} = E_{e,\mu}(\xi_{2n}^{\mu})$ and $E_{2n+1}^{\mu} = E_{o,\mu}(\xi_{2n+1}^{\mu})$. We have that

$$\begin{aligned} E_{2n}^{\mu} &= \frac{2^{\frac{1}{2}-\mu}}{\pi \Gamma(\mu + \frac{1}{2})} \int_{\mathbb{C}} \left| \frac{z^{2n}}{(\gamma_{\mu}(2n))^{\frac{1}{2}}} \right|^2 |z|^2 K_{\mu-\frac{1}{2}}(|z|^2) |z|^{2\mu+1} dx dy \\ &= \frac{2^{\frac{1}{2}-\mu} 2}{\Gamma(\mu + \frac{1}{2}) \gamma_{\mu}(2n)} \int_0^{\infty} K_{\mu-\frac{1}{2}}(r^2) r^{2(2n+\mu+2)} dr \\ &= \frac{2^{\frac{1}{2}-\mu}}{\Gamma(\mu + \frac{1}{2}) \gamma_{\mu}(2n)} \int_0^{\infty} K_{\mu-\frac{1}{2}}(s) s^{2n+\mu+\frac{3}{2}} ds. \end{aligned}$$

Since $2n + \mu + \frac{5}{2} > |\mu - \frac{1}{2}|$ we can use formula (2.7) to write

$$E_{2n}^{\mu} = \frac{2^{\frac{1}{2}-\mu}}{\Gamma(\mu + \frac{1}{2}) \gamma_{\mu}(2n)} 2^{2n+\mu+\frac{1}{2}} \Gamma\left(n + \frac{3}{2}\right) \Gamma(n + \mu + 1),$$

which simplifies (by using (2.1)) to

$$E_{2n}^{\mu} = \frac{2\Gamma(n + \frac{3}{2}) \Gamma(n + \mu + 1)}{\Gamma(n + 1) \Gamma(n + \mu + \frac{1}{2})}. \quad (6.1)$$

Similarly we have that

$$\begin{aligned} E_{2n+1}^{\mu} &= \frac{2^{\frac{1}{2}-\mu}}{\pi \Gamma(\mu + \frac{1}{2})} \int_{\mathbb{C}} \left| \frac{z^{2n+1}}{(\gamma_{\mu}(2n+1))^{\frac{1}{2}}} \right|^2 |z|^2 K_{\mu+\frac{1}{2}}(|z|^2) |z|^{2\mu+1} dx dy \\ &= \frac{2^{\frac{1}{2}-\mu} 2}{\Gamma(\mu + \frac{1}{2}) \gamma_{\mu}(2n+1)} \int_0^{\infty} K_{\mu+\frac{1}{2}}(r^2) r^{2(2n+\mu+3)} dr \\ &= \frac{2^{\frac{1}{2}-\mu}}{\Gamma(\mu + \frac{1}{2}) \gamma_{\mu}(2n+1)} \int_0^{\infty} K_{\mu+\frac{1}{2}}(s) s^{2n+\mu+\frac{5}{2}} ds. \end{aligned}$$

Since $2n + \mu + \frac{7}{2} > |\mu + \frac{1}{2}|$ we can use formula (2.7) to write

$$E_{2n+1}^{\mu} = \frac{2^{\frac{1}{2}-\mu}}{\Gamma(\mu + \frac{1}{2}) \gamma_{\mu}(2n+1)} 2^{2n+\mu+\frac{3}{2}} \Gamma\left(n + \frac{3}{2}\right) \Gamma(n + \mu + 2),$$

which simplifies (by using (2.2)) to

$$E_{2n+1}^\mu = \frac{2\Gamma\left(n + \frac{3}{2}\right)\Gamma(n + \mu + 2)}{\Gamma(n + 1)\Gamma\left(n + \mu + \frac{3}{2}\right)}. \quad (6.2)$$

When $\mu = 0$ formulas (6.1) and (6.2) become

$$E_{2n}^0 = 2n + 1 \quad \text{and} \quad E_{2n+1}^0 = 2n + 2,$$

which agrees with the known result that the (undeformed) energy E_n of the function ξ_n is $n + 1$ (see [Snt1]).

In [A-S.1] the following two reverse log-Sobolev inequalities are proved in the context of μ -deformed Segal-Bargmann analysis (Theorems 5.1 and 5.2).

Theorem 6.1 *For all $c > 1$ there exists a real number $P_e(c, \mu)$ such that for $f \in \mathcal{B}_{e,\mu}^2$ we have*

$$E_{e,\mu}(f) \leq cS_{L^2(\mathbb{C}, d\nu_{e,\mu})}(f) + P_e(c, \mu) \|f\|_{\mathcal{B}_{e,\mu}^2}.$$

Theorem 6.2 *For all $c > 1$ there exists a real number $P_o(c, \mu)$ such that for $f \in \mathcal{B}_{o,\mu}^2$ we have*

$$E_{o,\mu}(f) \leq cS_{L^2(\mathbb{C}, d\nu_{o,\mu})}(f) + P_o(c, \mu) \|f\|_{\mathcal{B}_{o,\mu}^2}.$$

A direct consequence of these results is the following reverse log-Sobolev inequality in the μ -deformed Segal-Bargmann space \mathcal{B}_μ^2 (Theorem 5.3 in [A-S.1]).

Theorem 6.3 *For all $c > 1$ there exists a real number $P(c, \mu)$ such that for $f \in \mathcal{B}_\mu^2$ we have*

$$E_\mu(f) \leq c\left(S_{L^2(\mathbb{C}, d\nu_{e,\mu})}(f_e) + S_{L^2(\mathbb{C}, d\nu_{o,\mu})}(f_o)\right) + P(c, \mu) \|f\|_{\mathcal{B}_\mu^2}.$$

In particular, if we consider the elements ξ_n^μ , $n = 0, 1, \dots$ of the canonical basis of \mathcal{B}_μ^2 , Theorem 6.1 tells us that for all $c > 1$ there exists a constant $P_e(c, \mu)$ such that for all $n = 0, 1, \dots$ we have that

$$E_{2n}^\mu \leq cS_{2n}^\mu + P_e(c, \mu), \quad (6.3)$$

and Theorem 6.2 tells us that for all $c > 1$ there exists a constant $P_o(c, \mu)$ such that for all $n = 0, 1, \dots$ we have that

$$E_{2n+1}^\mu \leq cS_{2n+1}^\mu + P_o(c, \mu). \quad (6.4)$$

Remark. By using Stirling's formula it is easy to see from (6.1) and (6.2) that for fixed $n = 0, 1, \dots$, we have that $E_n^\mu \rightarrow +\infty$ as $\mu \rightarrow +\infty$. We already

know that $S_{2n+1}^\mu \rightarrow -\infty$ as $\mu \rightarrow +\infty$ (see Theorem 4.2). Then (6.4) tells us that for any $c > 1$ and any $n = 0, 1, \dots$ we have that $P_o(c, \mu) \geq E_{2n+1}^\mu - cS_{2n+1}^\mu \rightarrow +\infty$ as $\mu \rightarrow +\infty$. That is, the values of the constant $P_o(c, \mu)$ in Theorem 6.2 will be as large as we want, by taking $\mu > 0$ large enough.

So Theorem 6.1 tells us that $c > 1$ is a sufficient condition to conclude the existence of the constant $P_e(c, \mu)$ such that the inequality (6.3) holds for all $n = 0, 1, \dots$. We will prove now that this condition is also necessary, by showing that for fixed $\mu > -\frac{1}{2}$, the sequence $\{E_{2n}^\mu - cS_{2n}^\mu\}_{n=0}^\infty$ is bounded above if and only if $c > 1$.

By using the formula

$$\frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} = z^{\alpha - \beta} (1 + O(z^{-1})), \quad (6.5)$$

valid for $|\arg z| < \pi$ and $z \rightarrow \infty$ (see [M-O-S], p. 12) we can write the following asymptotics for E_{2n}^μ :

$$\begin{aligned} E_{2n}^\mu &= 2 \frac{\Gamma(n + \frac{3}{2})}{\Gamma(n + 1)} \frac{\Gamma(n + \mu + 1)}{\Gamma(n + \mu + \frac{1}{2})} \\ &= 2n^{\frac{1}{2}} (1 + O(n^{-1})) n^{\frac{1}{2}} (1 + O(n^{-1})) \\ &= 2n + O(1). \end{aligned}$$

Also, by using (2.1), (5.4), (5.5) and formula (4.1) for S_{2n}^μ , we can write

$$\begin{aligned} S_{2n}^\mu &= n \left(\psi\left(\mu + n + \frac{1}{2}\right) + \psi(n + 1) \right) - \log \frac{\Gamma(n + 1) \Gamma(\mu + n + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \\ &= n \left(\log\left(\mu + n + \frac{1}{2}\right) + \log(n + 1) + O(n^{-1}) \right) \\ &\quad - \left(n + \frac{1}{2}\right) \log(n + 1) + n + 1 \\ &\quad - (\mu + n) \log\left(\mu + n + \frac{1}{2}\right) + \mu + n + \frac{1}{2} + O(1) \\ &= -\frac{1}{2} \log(n + 1) - \mu \log\left(\mu + n + \frac{1}{2}\right) + 2n + O(1). \end{aligned}$$

Then we have that

$$\begin{aligned}
E_{2n}^\mu - cS_{2n}^\mu &= 2n + O(1) - c \begin{pmatrix} -\frac{1}{2} \log(n+1) \\ -\mu \log\left(\mu + n + \frac{1}{2}\right) \\ +2n + O(1) \end{pmatrix} \\
&= (1-c)2n + \frac{c}{2} \log(n+1) + c\mu \log\left(\mu + n + \frac{1}{2}\right) + O(1) \\
&= (1-c)2n + \frac{c}{2} \log \frac{n+1}{\mu + n + \frac{1}{2}} \\
&\quad + c \left(\mu + \frac{1}{2}\right) \log\left(\mu + n + \frac{1}{2}\right) + O(1) \\
&= (1-c)2n + c \left(\mu + \frac{1}{2}\right) \log\left(\mu + n + \frac{1}{2}\right) + O(1).
\end{aligned}$$

Clearly, the sequence $\{E_{2n}^\mu - cS_{2n}^\mu\}_{n=0}^\infty$ is bounded above if and only if $c > 1$. This shows that the condition $c > 1$ is the best possible in the reverse log-Sobolev inequality in Theorem 6.1, namely that this inequality does not hold for $c \leq 1$.

Now let us consider Theorem 6.2. We know that $c > 1$ is a sufficient condition to conclude the existence of the constant $P_o(c, \mu)$ such that the inequality (6.4) holds for all $n = 0, 1, \dots$. We will see now that this condition is also necessary, by showing that for fixed $\mu > -\frac{1}{2}$, the sequence $\{E_{2n+1}^\mu - cS_{2n+1}^\mu\}_{n=0}^\infty$ is bounded above if and only if $c > 1$.

By using (6.5) we can write the following asymptotics for E_{2n+1}^μ :

$$\begin{aligned}
E_{2n+1}^\mu &= \frac{2\Gamma\left(n + \frac{3}{2}\right)\Gamma(n + \mu + 2)}{\Gamma(n+1)\Gamma\left(n + \mu + \frac{3}{2}\right)} \\
&= 2n^{\frac{1}{2}}(1 + O(n^{-1}))n^{\frac{1}{2}}(1 + O(n^{-1})) \\
&= 2n + O(1).
\end{aligned}$$

By using (2.2), (5.4), (5.5) and formula (4.3) for S_{2n+1}^μ , we can write

$$\begin{aligned}
S_{2n+1}^\mu &= \left(n + \frac{1}{2}\right) \left(\psi\left(\mu + n + \frac{3}{2}\right) + \psi(n+1)\right) \\
&\quad - \log \frac{\Gamma(n+1)\Gamma\left(\mu + n + \frac{3}{2}\right)}{\Gamma\left(\mu + \frac{1}{2}\right)} \\
&= \left(n + \frac{1}{2}\right) \left(\log\left(\mu + n + \frac{3}{2}\right) + \log(n+1) + O(n^{-1})\right) \\
&\quad - \left(n + \frac{1}{2}\right) \log(n+1) + n + 1 \\
&\quad - (\mu + n + 1) \log\left(\mu + n + \frac{3}{2}\right) + \mu + n + \frac{3}{2} + O(1) \\
&= -\left(\mu + \frac{1}{2}\right) \log\left(\mu + n + \frac{3}{2}\right) + 2n + O(1).
\end{aligned}$$

Then we have that

$$\begin{aligned} E_{2n+1}^\mu - cS_{2n+1}^\mu &= 2n + O(1) - c \left(-\left(\mu + \frac{1}{2}\right) \log \left(\mu + n + \frac{3}{2}\right) \right. \\ &\quad \left. + 2n + O(1) \right) \\ &= (1 - c) 2n + c \left(\mu + \frac{1}{2} \right) \log \left(\mu + n + \frac{3}{2} \right) + O(1). \end{aligned}$$

As in the case of Theorem 6.1 considered above, we see now that the sequence $\{E_{2n+1}^\mu - cS_{2n+1}^\mu\}_{n=0}^\infty$ is bounded above if and only if $c > 1$, which shows that the condition $c > 1$ is the best possible in the reverse log-Sobolev inequality in Theorem 6.2.

Either one of the two cases considered in this section shows that the condition $c > 1$ in Theorem 6.3 is also the best possible.

7 Final remarks

In conclusion, we have just a few comments.

Firstly, it would be interesting to evaluate in closed form the entropies of the elements of the canonical basis of $L^2(\mathbb{R}, dg_\mu)$. This has not even been done yet in the case $\mu = 0$.

Secondly, we would like to repeat the conjecture that the μ -deformed Segal-Bargmann transform increases the entropy of some functions. And again, this is plausible since it is known to be true when $\mu = 0$. (See [Snt1].)

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